EXERCISES 8.4

In Exercises 1 and 2, shift the summation indices to combine the given expressions into the sum of a finite number of terms and a single summation.

1. **(a)** $2\sum_{n=0}^{\infty} a_n x^{n+c} + (1+x)\sum_{n=0}^{\infty} a_n x^{n+c-2}$.

(b) $3\sum_{n=0}^{\infty} a_n x^{n+c} + 2x^2\sum_{n=0}^{\infty} a_n x^{n+c-1}$.

2. **(a)** $(x-x^3)\sum_{n=0}^{\infty} a_n x^{n+c} + 3\sum_{n=0}^{\infty} a_n x^{n+c-1}$.

(b) $(x^2-x)\sum_{n=0}^{\infty} a_n x^{n+c} + 2\sum_{n=0}^{\infty} a_n x^{n+c-2}$.

In Exercises 3 through 6, use long division and multiplication of series to find the first four terms of the given expressions.

3. **(a)** $\dfrac{1}{\sum_{n=0}^{\infty}(-1)^n x^n/(n+1)}$.

(b) $(1 - x/2 + x^2/4 - x^3/8 + x^4/16 - x^5/32 + \cdots)\exp(x)$.

(c) $(1 - x/2 + x^2/3 - x^3/4 + x^4/5 - \cdots)(1 - x + x^2/2 - x^3/3 + x^4/4 - \cdots)$.

4. **(a)** $(1+2x+x^2)/(3-x+2x^4)$.

(b) $\left(\sum_{n=1}^{\infty}\dfrac{x^n}{n^2}\right)\left(\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}x^n}{(n+1)}\right)$.

5. **(a)** $\displaystyle\int \frac{1}{x}\left(\frac{1-3x+x^2}{2-\exp(x)}\right)dx$. **(b)** $\displaystyle\int \frac{\exp x}{(x+x^2+x^3)}dx$.

6. **(a)** $\displaystyle\int \frac{1}{x^2}\frac{(1+2x-x^2)}{(1+x+2x^3)}dx$. **(b)** $\displaystyle\int \frac{1}{x}\frac{\exp(-x)}{(1-2x+2x^2)}dx$.

In Exercises 7 through 26, find two linearly independent solutions for $x > 0$, and determine at least the first four leading terms in the second solution $y_2(x)$.

7. $4x^2y'' + 2xy' + (x-2)y = 0$.
8. $3x^2y'' - xy' + (x+1)y = 0$.
9. $2x^2y'' + xy' - (2x+1)y = 0$.
10. $2x^2y'' + xy' - (3x+1)y = 0$.
11. $(x^2-1)y'' + 2xy' + y = 0$.
12. $2x^2y'' + 2xy' + (x^2-2)y = 0$.
13. $x(1-x)y'' + (1-x)y' - y = 0$.
14. $2x^2y'' - 2xy' + (x^2+2)y = 0$.
15. $x^2y'' + (2x^2-x)y' + y = 0$.
16. $x^2y'' + 2(x^2-x)y' + 2y = 0$.
17. $x^2y'' + (x^2-2x)y' + 2y = 0$.
18. $x^2y'' - xy' + (x^2+1)y = 0$.
19. $16x^2y'' + 8xy' + (16x+1)y = 0$.
20. $2x^2y'' + 2xy' + (x-2)y = 0$.
21. $x^2y'' + (x^2-x)y' - 3y = 0$.
22. $4x^2y'' - 2x^2y' + (2x+1)y = 0$.
23. $x^2y'' + (x^2+x)y' - 4y = 0$.
24. $9x^2y'' - 6xy' + 2y = 0$.
25. $x^2y'' - 4xy' + 20y = 0$.
26. $4x^2y'' + 8xy' + 5y = 0$.

27. By shifting the critical point to the origin, find two linearly independent solutions of the following equation in an interval of the form $0 < x+1 < d$:

$$2(x+1)y'' + y' - (x+1)y = 0.$$

28. By shifting the critical point to the origin, find two linearly independent solutions of the following equation in an interval of the form $0 < x-2 < d$:

$$(x-2)^2y'' - (x-2)y' + (x^2-4x+5)y = 0.$$

8.5 The Gamma Function Revisited

more about the Gamma function

The function $\Gamma(x)$, called the **gamma function**, was introduced in (4) of Section 7.1 in connection with the Laplace transform of t^a when a is not an integer, and it was defined in terms of the improper integral

$$\Gamma(x) = \int_0^{\infty} e^{-t}t^{x-1}\,dt \quad \text{for } x > 0. \tag{32}$$

a fundamental result

It was shown that $\Gamma(x)$ satisfies the recurrence relation

$$\Gamma(x+1) = x\Gamma(x) \quad \text{for } x > 0, \tag{33}$$

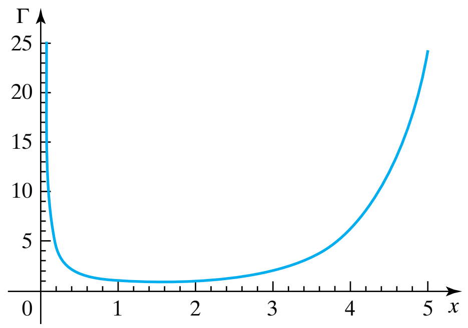


FIGURE 8.4 The function $\Gamma(x)$ in the interval $0 < x < 5$.

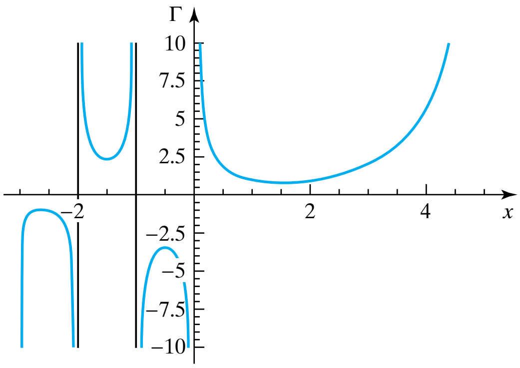


FIGURE 8.5 The function $\Gamma(x)$ in the interval $-3 < x < 4$.

and that when x is a positive integer n the gamma function reduces to

$$\boxed{\Gamma(n+1) = n!.} \tag{34}$$

Thus, for any real $x > 0$, the function $\Gamma(x)$ interpolates continuously between successive values of $n!$, and so generalizes the factorial function to nonintegral values of n. For obvious reasons the gamma function is sometimes called the **factorial function**. Figure 8.4 shows a graph of $\Gamma(x)$ in the interval $0 < x < 5$.

The gamma function can be extended to $x < 0$ for $x \neq -1, -2, \ldots,$ at which point it becomes infinite. A graph of $\Gamma(x)$ in the interval $-3 < x < 4$ is shown in Fig. 8.5.

The value of $\Gamma(1/2)$ is often needed, and it can be found by means of the following method in which the integral defining $\Gamma(1/2)$ is squared and converted to a double integral that is easily evaluated. If the method used is unfamiliar the details can be omitted, though the result given in (35) is useful and should be remembered.

From (32) we have

$$[\Gamma(1/2)]^2 = \left(\int_0^\infty u^{-1/2} e^{-u} du\right)\left(\int_0^\infty v^{-1/2} e^{-v} dv\right),$$

where the two dummy variables u and v have been introduced to avoid confusion when the product of integrals is combined.

Writing $u = x^2$ and $v = y^2$ allows this product of integrals to be written as

$$[\Gamma(1/2)]^2 = \left(\int_0^\infty 2e^{-x^2} dx\right)\left(\int_0^\infty 2e^{-y^2} dy\right) = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy.$$

As the integral in terms of cartesian coordinates is only evaluated over the first quadrant, changing to the polar coordinates (r, θ) by setting $x = r\cos\theta$, $y = r\sin\theta$, and using the result $r^2 = x^2 + y^2$ reduces this last integral to

$$[\Gamma(1/2)]^2 = \lim_{\rho\to\infty} 4\int_0^{\pi/2} d\theta \int_0^\rho e^{-r^2} r\,dr = 4\cdot(\pi/2)\lim_{\rho\to\infty}\left[-\frac{1}{2}e^{-r^2}\right]_0^\rho = \pi.$$

a useful special case

Taking the square root shows that

$$\boxed{\Gamma(1/2) = \sqrt{\pi}.} \tag{35}$$

When x is a multiple of $1/2$, repeated use of recurrence relation (33) combined with result (35) allows $\Gamma(x)$ to be simplified, as illustrated in the following example.

EXAMPLE 8.13 Find (a) $\Gamma(7/2)$ and (b) $\Gamma(-3/2)$.

Solution

(a) From (33) it follows that

$$\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{15}{8}\sqrt{\pi}.$$

(b) Setting $x = -3/2$ in (33) gives

$$\left(-\frac{3}{2}\right)\Gamma\left(-\frac{3}{2}\right) = \Gamma\left(-\frac{1}{2}\right),$$

whereas setting $x = -1/2$ in (33) gives

$$\left(-\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right) = \Gamma(1/2) = \sqrt{\pi}.$$

So, combining these two results, we find that

$$\Gamma\left(-\frac{3}{2}\right) = \left(-\frac{2}{3}\right)\left(-\frac{2}{1}\right)\Gamma(1/2) = \frac{4}{3}\sqrt{\pi}.$$

The reason for this re-examination of the gamma function is because it enables the coefficients of a series expansion to be expressed in a concise form. For example, it follows directly from (34) that the binomial coefficient

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} = \frac{\Gamma(n+1)}{\Gamma(m+1)\Gamma(n-m+1)}. \tag{36}$$

Expressing a binomial coefficient with integer entries in terms of the gamma function offers no particular advantage over the use of factorials, but the preceding result generalizes to the more useful result

$$\binom{\alpha}{m} = \frac{\Gamma(\alpha+1)}{\Gamma(m+1)\Gamma(\alpha-m+1)} \tag{37}$$

when α is any nonnegative real number (not necessarily an integer). This expression is often useful when performing numerical calculations.

As another example of the use of (33) we notice that we can write

$$a(a+1)(a+2)\ldots(a+n) = \frac{\Gamma(a+n+1)}{\Gamma(a)}, \tag{38}$$

where n is a positive integer and the real number $a > 0$. Thus, for example, in terms of the gamma function the following product becomes

$$\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\left(\frac{7}{2}\right) = \frac{\Gamma\left(\frac{1}{2}+3+1\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{\Gamma\left(\frac{9}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}.$$

Result (38) generalizes further to provide a concise representation of the product of $n+1$ factors $c(c+d)(c+2d)\ldots(c+nd)$. By writing the product as

$$c(c+d)(c+2d)\ldots(c+nd) = d^{n+1}\left(\frac{c}{d}\right)\left(\frac{c}{d}+1\right)\left(\frac{c}{d}+2\right)\cdots\left(\frac{c}{d}+n\right),$$

and then setting $a = c/d$ in (38), we arrive at the useful result

$$c(c+d)(c+2d)\ldots(c+nd) = d^{n+1}\frac{\Gamma\left(\frac{c}{d}+n+1\right)}{\Gamma\left(\frac{c}{d}\right)}. \tag{39}$$

EXAMPLE 8.14

The nth coefficient of a series is given by

$$a_n = \frac{1\cdot 5\cdot 9\cdot 13\ldots(4n+1)}{2^n}.$$

Express a_n in terms of the gamma function.

Solution Comparing the numerator of a_n with result (39) shows that it contains $n+1$ factors, and in the notation of (39) we have $c = 1$ and $d = 4$. Thus,

$$1\cdot 5\cdot 9\cdot 13\ldots(4n+1) = 4^{n+1}\frac{\Gamma\left(n+\frac{5}{4}\right)}{\Gamma\left(\frac{1}{4}\right)},$$

so dividing by 2^n we find that

$$a_n = 4^{n+1}\frac{\Gamma\left(n+\frac{5}{4}\right)}{2^n\Gamma\left(\frac{1}{4}\right)} = 2^{n+2}\frac{\Gamma\left(n+\frac{5}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}.$$ ■

Two special products of this type arise when working with series as, for example, occurs in the case of Legendre polynomials. These products involve either the product of consecutive pairs of odd numbers or the product of consecutive pairs of even numbers. Although these products can be expressed in terms of the gamma function, a convenient and concise **double factorial** notation is used. We define the double factorial !! as follows:

the double factorial

$$1\cdot 3\cdot 5\cdots(2n+1) = (2n+1)!! \quad \text{and} \quad 2\cdot 4\cdot 6\cdots(2n) = (2n)!!. \tag{40}$$

Alternative expressions for these double factorials in terms of the usual factorial function are

$$(2n+1)!! = \frac{(2n+1)!}{2^n n!} \quad \text{and} \quad (2n)!! = 2^n n!. \tag{41}$$

The following relationship connecting gamma functions is sometimes useful:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}. \tag{42}$$

However, this result will not be proved here as it requires the techniques of complex integration.

the beta function

In passing, we mention a function $B(x, y)$ called the *beta function* that is related to the gamma function. The **beta function**, which has applications in statistics and elsewhere, is defined as the integral

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \quad \text{with} \quad x > 0, y > 0. \tag{43}$$

The following are the most important properties of the beta function:

Symmetry:

$$B(x, y) = B(y, x) \tag{44}$$

relating gamma and beta functions

Connection with the gamma function:

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{45}$$

Relationship between beta functions:

$$B(x, y) = \left(\frac{y-1}{x+y-1}\right) B(x, y-1) = \left(\frac{x+y}{y}\right) B(x, y+1), \tag{46}$$

Special values:

$$B(1, 1) = 1 \quad \text{and} \quad B(1/2, 1/2) = \pi. \tag{47}$$

Outline proofs of results (42) to (44) will be found in the harder exercises at the end of this section.

The gamma function in the complex plane is discussed in reference [6.7], and general information about the gamma function and related functions is contained in Chapter 6 of reference [G.1] and Chapter 11 of reference [G.3].

Summary

The gamma function that was introduced earlier was seen to provide a natural extension to arbitrary values of x of the factorial function $n!$, where n is an integer. In this section the gamma function was examined in greater detail and some useful values were derived in terms of π. The beta function was then defined and related to the gamma function.

EXERCISES 8.5

1. Express $\Gamma(5/2), \Gamma(-5/2)$, and $\Gamma(9/2)$ in terms of $\sqrt{\pi}$.
2. Express $\Gamma(-9/2), \Gamma(11/2)$, and $\Gamma(-11/2)$ in terms of $\sqrt{\pi}$.
3. Express $\Gamma(5/4), \Gamma(-5/4)$, and $\Gamma(7/4)$ in terms of either $\Gamma(1/4)$ or $\Gamma(-1/4)$.
4. Express $\Gamma(-7/4), \Gamma(9/4)$, and $\Gamma(3/4)$ in terms of either $\Gamma(1/4)$ or $\Gamma(-1/4)$.
5. Express the product $6 \cdot 11 \cdot 16 \cdot 21 \ldots (5n+6)$ in terms of the gamma function.
6. Express the product $1 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \ldots .(2n+1)$ in terms of the gamma function.

7. Express the product $5 \cdot 8 \cdot 11 \cdot 14 \ldots (3n+5)$ in terms of the gamma function.

8. Express the product $4 \cdot 8 \cdot 12 \cdot 16 \ldots (4n+4)$ in terms of the gamma function.

9. Show that

$$\Gamma\left(\frac{1}{2}-n\right) = \frac{(-1)^n\sqrt{\pi}}{\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\cdots\left(\frac{1}{2}\right)}.$$

10. Show that

$$\Gamma\left(n+\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\left(\frac{1}{2}\right)\sqrt{\pi}.$$

The following slightly harder exercises provide more information about the gamma function.

11.* Use the result $\Gamma(n+\frac{1}{2}) = (n-\frac{1}{2})\Gamma(n-\frac{1}{2})$ with the result of Exercise 9 to show that

$$\Gamma(2n) = \frac{2^{2n-1}\Gamma(n)\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}}.$$

12.* Show that $\Gamma(x) = \int_0^1 (\ln \frac{1}{u})^{x-1} du$ for $x > 0$.

13.* Show that $\Gamma(x) = 2\int_0^\infty e^{-u^2} u^{2x-1} du$ for $x > 0$.

14.* The function $\psi(x)$, called the **psi function** or the **digamma function**, is defined as

$$\psi(x) = \frac{d}{dx}[\ln \Gamma(x)].$$

Show that

$$\psi(x+1) = \psi(x) + \frac{1}{x} \quad \text{for } x > 0.$$

15.* Use the result of Exercise 14 to show that

$$\psi(x+n) = \psi(x) + \sum_{k=0}^{n-1} \frac{1}{x+k} \text{ where } n > 1 \text{ is an integer.}$$

16.* By making the variable change $u = 1 - t$ in the integral defining $B(x, y)$, show that $B(x, y) = B(y, x)$.

17.* Integrate $B(x, y)$ by parts to obtain the result of (46) that

$$B(x, y) = \left(\frac{y-1}{x+y-1}\right) B(x, y-1),$$

and use this result to obtain the second result of (46).

18.* Use the result of Exercise 17 to show that if m and n are integers,

$$B(m, n) = \frac{(m-n)!(n-1)!}{(m+m-1)!} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},$$

and so

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

8.6 Bessel Function of the First Kind $J_n(x)$

Bessel's equation

In standard form, **Bessel's equation** is written

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0, \tag{48}$$

where $\nu \geq 0$ is a real number. Another useful form of Bessel's equation that often arises in applications is

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2x^2 - \nu^2)y = 0. \tag{49}$$

This form of the equation is obtained from (48) by first making the change of variable $x = \lambda u$, and then replacing u by x.

When developing the properties of Bessel functions in this section the standard form of the equation given in (48) will be used. Applications of Bessel functions to partial differential equations are made in Chapter 18.

Bessel's equation has a singularity at the origin, and using the notation of Section 8.4 with $P(x) = 1/x$ and $Q(x) = (x^2 - \nu^2)/x^2$, we find that

$$p_0 = \lim_{x\to 0} xP(x) = 1 \quad \text{and} \quad q_0 = \lim_{x\to 0} x^2Q(x) = -\nu^2,$$

showing that the origin is a regular singular point.

The indicial equation is seen to be

$$c^2 - \nu^2 = 0, \tag{50}$$

so the roots $c_1 = \nu$ and $c_2 = -\nu$ are distinct when $\nu \neq 0$, and there is a repeated zero root when $\nu = 0$. Thus, when $\nu = 0$, the second Frobenius solution will contain a logarithmic term, whereas when $c_1 - c_2$ is an integer the second Frobenius solution may or may not contain a logarithmic term. When $c_1 - c_2 \neq 0$ is not an integer, neither of the two linearly independent Frobenius solutions contains a logarithmic term.

Substituting $y(x) = \sum_{r=0}^{\infty} a_r x^{r+c}$ into (48) gives

$$\sum_{r=0}^{\infty}(r+c)(r+c-1)a_r x^{r+c} + \sum_{r=0}^{\infty}(r+c)a_r x^{r+c} + \sum_{r=0}^{\infty} a_r x^{r+c+2} - \nu^2 \sum_{r=0}^{\infty} a_r x^{r+c} = 0.$$

Shifting the summation index in the third summation and collecting terms under a single summation leads to the result

$$(c^2 - \nu^2)a_0 x^c + [(c+1)^2 - \nu^2]a_1 x^{c+1} + \sum_{r=2}^{\infty}[(r+c+\nu)(r+c-\nu)a_r + a_{r-2}]x^{r+c} = 0.$$

Equating the coefficients of powers of x to zero shows the following:

Coefficient of x^c:

$$(c^2 - \nu^2)a_0 = 0 \quad \text{(the indicial equation, because } a_0 \neq 0)$$

Coefficient of x^{c+1}:

$$[(c+1)^2 - \nu^2]a_1 = 0 \quad \text{(a condition on } a_1)$$

Coefficient of x^{r+c}:

$$[(r+c)^2 - \nu^2]a_r + a_{r-2} = 0 \quad \text{(a recurrence relation)} \tag{51}$$

As $(c+1)^2 - \nu^2 \neq 0$, it follows from the second result that $a_1 = 0$, and then from the recurrence relation (51) that $a_r = 0$ for all odd r. As only even indices r are involved in the recurrence relation, we set $r = 2m$ with $m = 0, 1, \dots,$ after which substituting $c = \nu$ in the recurrence relation reduces it to

$$a_{2m} = -\frac{1}{4m(m+\nu)}a_{2m-2}, \quad \text{for } m = 1, 2, \dots. \tag{52}$$

As a_0 is arbitrary, we normalize the solution in the standard manner by setting

$$a_0 = \frac{1}{2^\nu \Gamma(1+\nu)},$$

after which the coefficients a_{2m} become

$$a_2 = -\frac{a_0}{2^2(1+\nu)} = -\frac{1}{2^{2+\nu}1!\Gamma(2+\nu)}, \quad a_4 = -\frac{a_2}{2^2 2(2+\nu)} = \frac{1}{2^{4+\nu}2!\Gamma(3+\nu)}, \dots,$$

and, in general,

$$a_{2m} = -\frac{(-1)^m}{2^{2m+\nu}m!\Gamma(m+1+\nu)}, \quad \text{for } m = 1, 2, \dots. \tag{53}$$

the Bessel function $J_\nu(x)$

Using this result in the first Frobenius solution, which hereafter will be denoted by $J_\nu(x)$ and called a **Bessel function of the first kind of order** ν, gives

$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m!\Gamma(m+1+\nu)} \quad \text{for } x \geq 0. \tag{54}$$

When $x < 0$ the corresponding expression for $J_\nu(x)$ follows from the preceding result by reversing the sign of x in the series and replacing x^ν by $|x|^\nu$. The ratio test shows the series for $J_\nu(x)$ to be absolutely convergent for all x.

So far ν has been an arbitrary nonnegative number, but the standard convention is that when ν is an integer it is denoted by n. Using the result that when $\nu = n$ the gamma function $\Gamma(m+1+n) = (m+n)!$ allows $J_n(x)$ to be written in the simpler form

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+n}}{2^{2m+n} m!(m+n)!}, \quad \text{for } n = 0, 1, 2, \ldots. \tag{55}$$

It was because of this use of n that, to avoid confusion, the summation index in the series was chosen to be m. The two most important special cases of (55) are:

Bessel functions $J_0(x)$ and $J_1(x)$

Bessel function of the first kind of order zero:

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \cdots \tag{56}$$

Bessel function of the first kind of order 1:

$$J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1} m!(m+1)!} = \frac{x}{2} - \frac{x^3}{2^3 1!2!} + \frac{x^5}{2^5 2!3!} - \frac{x^7}{2^7 3!4!} + \cdots. \tag{57}$$

Graphs of $J_0(x)$, $J_1(x)$, and $J_2(x)$ are shown in Fig. 8.6.

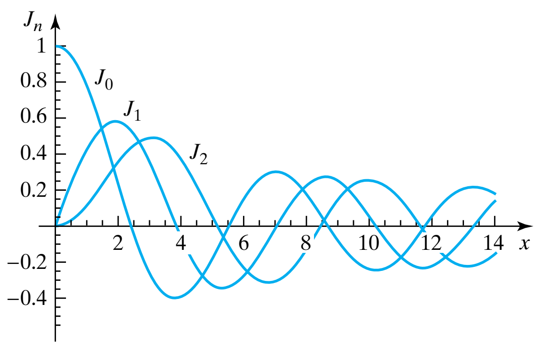


FIGURE 8.6 Graphs of the Bessel functions of the first kind $J_0(x)$, $J_1(x)$, and $J_2(x)$.

Having found $J_\nu(x)$, which is one solution of Bessel's equation (48), we must now find a second linearly independent solution in order to arrive at a basis for solutions of the equation, and hence to arive at the general solution. The nature of a second linearly independent solution will depend on the value of ν, and the simplest situation arises when ν is not an integer. In this case, because $c^2 = \nu^2$, a second linearly independent solution will follow from (54) by replacing ν by $-\nu$. Denoting this second solution by $J_{-\nu}(x)$ we find that

$$J_{-\nu}(x) = |x|^{-\nu} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-\nu} m! \Gamma(m+1-\nu)} \quad \text{for } x \neq 0. \tag{58}$$

general solution of Bessel's equation

When ν is *not* an integer, the **general solution of Bessel's equation** (48) can be written

$$y(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x), \quad \text{for } x \neq 0, \tag{59}$$

with C_1 and C_2 arbitrary constants. The corresponding general solution of (49) is then

$$y(x) = C_1 J_\nu(\lambda x) + C_2 J_{-\nu}(\lambda x), \quad \text{for } x \neq 0. \tag{60}$$

The nature of the second linearly independent solution when $\nu = n$ will be considered later. In the meantime we will show that when $\nu = n$, the Bessel functions $J_n(x)$ and $J_{-n}(x)$ are linearly dependent. This is most easily seen by taking the limit of (58) as $\nu \to n$. Gamma functions with negative integer arguments are infinite, so the coefficients a_{2m} in which they occur will all vanish, causing the summation to start at the value $m = n$. Using the result $\Gamma(m+1-n) = (m-n)!$ then shows that the series for $J_{-n}(x)$ is

$$J_{-n}(x) = \sum_{m=n}^{\infty} \frac{(-1)^m x^{2m-n}}{2^{2m-n} m!(m-n)!},$$

and after a shift of the summation index this becomes

$$J_{-n}(x) = (-1)^n \sum_{m=n}^{\infty} \frac{(-1)^m x^{2m+n}}{2^{2m+n} m!(m+n)!}, \quad \text{for } n = 1, 2, \ldots. \tag{61}$$

A comparison of (55) and (61) shows that $J_{-n}(x)$ is a constant multiple of $J_n(x)$, so the two functions $J_n(x)$ and $J_{-n}(x)$ are linearly dependent. To be precise,

$$J_{-n}(x) = (-1)^n J_n(x) \quad \text{for } n = 1, 2, \ldots. \tag{62}$$

The absolute convergence of the series for $J_\nu(x)$ allows it to be differentiated term by term. Using this fact, and comparing of the derivative of the series for $J_0(x)$ with the series for $J_1(x)$, shows that

$$J_0'(x) = -J_1(x). \tag{63}$$

This result is the simplest example of the many relationships that exist between Bessel functions. The four most important results are the following:

relationships between derivatives and some recurrence relations

Relationships between derivatives of $J_\nu(x)$:

$$\frac{d}{dx}[x^\nu J_\nu(x)] = x^\nu J_{\nu-1}(x) \tag{64}$$

$$\frac{d}{dx}\left[x^{-\nu} J_\nu(x)\right] = -x^{-\nu} J_{\nu-1}(x) \tag{65}$$

Recurrence relations involving $J_\nu(x)$:

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) \tag{66}$$

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_\nu'(x) \tag{67}$$

We show next that these results are easily verified by substituting the series solution for $J_\nu(x)$ given in (54) into each relationship, though the direct derivation of these relationships is a more complicated matter. An indication of one way in which to arrive at these results without appealing to the series solution (54) is to be found in the set of exercises at the end of this section.

To establish (64) we start by multiplying the series (54) for $J_\nu(x)$ by x^ν to obtain

$$x^\nu J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+2\nu}}{2^{2m+\nu} m!\Gamma(m+1+\nu)}.$$

Differentiating this result and removing a factor x^ν from the summation gives

$$\frac{d}{dx}[x^\nu J_\nu(x)] = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+\nu-1}}{2^{2m+\nu-1}\Gamma(m+\nu)},$$

but the series on the right-hand side is simply $J_{\nu-1}(x)$, so we have shown that

$$\frac{d}{dx}[x^\nu J_\nu(x)] = x^\nu J_{\nu-1}(x).$$

Result (65) is established in similar fashion by differentiating $x^{-\nu} J_\nu(x)$.

The recurrence relations can be obtained as follows. Carrying out the indicated differentiations and cancelling a factor x^ν in (64) and (65) gives

$$J_\nu'(x) = J_{\nu-1}(x) - \frac{\nu}{x} J_\nu(x) \tag{64$'$}$$

and

$$J_\nu'(x) = \frac{\nu}{x} J_\nu(x) - J_{\nu+1}(x). \tag{65$'$}$$

Results (66) and (67) now follow first by subtraction and then by addition of these two results.

Result (66) is useful because it relates $J_\nu(x)$ to $J_{\nu-1}(x)$ and $J_{\nu+1}(x)$, whereas (64) and (65) can be used to evaluate certain integrals involving $J_\nu(x)$, because by integrating (64) and (65) we obtain

$$\int x^\nu J_{\nu-1}(x)\mathrm{d}x = x^\nu J_\nu(x) + C \tag{68}$$

and

$$\int x^{-\nu} J_{\nu+1}(x)\mathrm{d}x = -x^{-\nu} J_\nu(x) + C. \tag{69}$$

EXAMPLE 8.15 Express $J_4(x)$ in terms of $J_0(x)$ and $J_1(x)$, and use the result to compute $J_4(6.2)$ given that $J_0(6.2) = 0.20175$ and $J_1(6.2) = -0.23292$.

Solution Rearranging (66) gives

$$J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) - J_{\nu-1}(x),$$

so setting $\nu = 3, 2,$ and 1 we have

$$J_4(x) = \frac{6}{x} J_3(x) - J_2(x), \quad J_3(x) = \frac{4}{x} J_2(x) - J_1(x), \text{ and } J_2(x) = \frac{2}{x} J_1(x) - J_0(x).$$

Eliminating $J_2(x)$ and $J_3(x)$ between these results gives the required expression

$$J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x).$$

Setting $x = 6.2$ and substituting the given values of $J_0(6.2)$ and $J_1(6.2)$ shows that $J_4(6.2) = 0.32941$. ■

Numerical values of Bessel functions are extensively tabulated, and subroutines that enable their calculation for arbitrary values of their argument are found in most computer algebra packages. See the references at the end of the chapter for some of the most extensive tabulations of Bessel functions.

EXAMPLE 8.16 Evaluate

$$\int \left(x^2 + \frac{1}{x}\right) J_1(x)dx.$$

Solution We write the integral as the sum of integrals

$$\int \left(x^2 + \frac{1}{x}\right) J_1(x)dx = \int x^2 J_1(x)dx + \int x^{-1} J_1(x)dx$$

and consider each separately. Setting $\nu = 2$ in (64) shows that

$$\frac{d}{dx}[x^2 J_2(x)] = x^2 J_1(x),$$

so it follows at once that

$$\int x^2 J_1(x)dx = x^2 J_2(x) + C.$$

The second integral is a little harder and requires the use of integration by parts. Writing it as

$$\int x^{-1} J_1(x)dx = \int x^{-2}[x J_1(x)]dx,$$

and noticing from (63) with $\nu = 1$ that $[xJ_1(x)]' = xJ_0(x)$, we find that

$$\int x^{-1}J_1(x)dx = \int x^{-2}[xJ_1(x)]dx = -J_1(x) + \int x^{-1}xJ_0(x)dx,$$

and so

$$\int x^{-1}J_1(x)dx = -J_1(x) + \int J_0(x)dx.$$

No further simplification is possible because $\int J_0(x)\mathrm{d}x$ cannot be expressed in terms of simpler functions, though $\int_0^x J_0(u)du$ is available in tabular form and it is easily evaluated numerically on a computer. However, we will see later that $\int_0^\infty J_n(x)dx = 1$ for $n = 0, 1, 2, \ldots$. ■

EXAMPLE 8.17 Evaluate $\int x^3 J_0(x)\mathrm{d}x$.

Solution Writing the integrand as the product $x^3J_0(x) = x^2[xJ_0(x)]$ and using (64) with $\nu = 1$ gives

$$\int x^3J_0(x)\mathrm{d}x = \int x^2[xJ_0(x)]\mathrm{d}x = \int x^2\frac{d}{dx}[xJ_1(x)]dx.$$

Integration by parts then gives

$$\int x^3J_0(x)\mathrm{d}x = x^3J_1(x) - 2x^2J_2(x) + C.$$ ■

It can be seen from Fig. 8.6 that the Bessel functions $J_0(x)$, $J_1(x)$, and $J_2(x)$ are oscillatory in nature and resemble damped sinusoids. The recurrence relation (66) implies that this same oscillatory property is true for all $J_n(x)$. Although these Bessel functions are not strictly periodic, in the sense that for any given n the zeros of $J_n(x)$ are not equally spaced along the x-axis, it can be shown that for fixed ν and large x the function $J_n(x)$ can be approximated by

$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \tag{70}$$

where the symbol $\sim$ is to be read "is asymptotically equal to," with the understanding that the term *asymptotic* is used here in the technical sense and means that the ratio of the two sides of the expression tends to 1 as $x \to \infty$. This last result is an example of what is called an **asymptotic expansion** of the function $J_\nu(x)$, and asymptotic expansions have the property that the larger x becomes, the more accurate the asymptotic expansion becomes.

asymptotic expansion of $J_\nu(x)$

When the Bessel functions $J_\nu(x)$ are required in a computer program, the series solution (54) is used for small x, and different approximations are used for large x and in the intermediate region between small and large x. Corresponding approximations are used when the order ν of a Bessel function is large. The simplest approximation to $J_\nu(x)$ for small x, which follows from (54) by setting $m = 0$, is

$$J_\nu(x) \approx \frac{1}{\Gamma(1+\nu)}\left(\frac{x}{2}\right)^\nu. \tag{71}$$

The fact that the series for $J_\nu(x)$ is an alternating series means that the maximum magnitude of the error made when the series is truncated after n terms is the absolute

TABLE 8.1 Zeros $j_{n,r}$ of $J_n(x)$ for $n = 0, 1, 2, 3$

r	$j_{0,r}$	$j_{1,r}$	$j_{2,r}$	$j_{3,r}$
1	2.40482	3.83171	5.13162	6.38016
2	5.52007	7.01559	8.41724	9.76102
3	8.65372	10.17347	11.61984	13.01520
4	11.79153	13.32369	14.79595	16.22347
5	14.93091	16.47063	17.95982	19.40942
6	18.07106	19.61586	21.11700	22.58273

value of the $(n+1)$th term. So, if the series

$$J_0(x) = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \cdots$$

is truncated after the term in x^4, the maximum error made is $|-x^6/[2^6(3!)^2]| = x^6/[2^6(3!)^2]$. Consequently, if $J_0(x)$ is approximated by

$$J_0(x) \approx 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2},$$

then in the interval $0 \le x \le a$, the absolute value of the maximum error will not exceed $a^6/[2^6(3!)^2]$. When $J_\nu(x)$ is required to be accurate to a given number of decimal places in an interval $0 \le x \le a$, this simple estimate determines how many terms must be retained in the series approximation for $J_\nu(x)$.

zeros of Bessel functions $J_n(x)$

When using Bessel functions in applications, it is often necessary to know the location of the zeros of $J_n(x)$, so for future reference Table 8.1 lists the first six zeros of $J_n(x)$ for $n = 0, 1, 2, 3$. In the table the rth zero of $J_n(x)$ is denoted by $j_{n,r}$, where the first suffix indicates the order of the Bessel function and the second suffix the number of the zero. As $J_n(0) = 0$ for $n \ge 1$, the zeros $j_{1,r}$, $j_{2,r}$, and $j_{3,r}$ have been numbered so the first entry to appear in each column is the first nonvanishing zero of the function involved. Thus, although $J_1(0) = 0$, the first entry to appear in the column for $j_{1,r}$ is 3.83171, which it will be seen from Fig. 8.6 is the first nonvanishing zero of $J_1(x)$.

Bessel Functions $J_{\pm n/2}(x)$

The Bessel functions $J_{\pm n/2}(x)$ are particularly simple, despite the fact that the difference between the indices $c_1 = n/2$ and $c_2 = -n/2$ is an integer. The easiest way to find the form of $J_{\pm n/2}(x)$ is to use the reduction to standard form given in Lemma 6.1 of Section 6.3 to remove the first derivative term from Bessel's equation.

It follows from the lemma that the substitution $u = x^{1/2}y$ reduces Bessel's equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

to the standard form for a second order equation

$$u'' + \left(1 - \frac{4\nu^2 - 1}{4x^2}\right)u = 0.$$

If we now consider the cases of $J_{1/2}(x)$ and $J_{-1/2}(x)$, corresponding to $\nu^2 = 1/4$, the differential equation simplifies to

$$u'' + u = 0,$$

with the general solution

$$u(x) = C_1 \sin x + C_2 \cos x.$$

As $y = x^{-1/2}u$, the general solution of Bessel's equation of order $\pm 1/2$ becomes

$$y(x) = C_1\sqrt{\frac{1}{x}}\sin x + C_2\sqrt{\frac{1}{x}}\cos x.$$

The two functions in the general solution for $y(x)$ are linearly independent, so we take for the solutions forming a basis for the differential equation with $\nu = \pm 1/2$ the functions $J_{1/2}(x)$ and $J_{-1/2}(x)$ given by

Bessel functions of fractional order

$$J_{1/2}(x) = C_1\sqrt{\frac{1}{x}}\sin x \quad \text{and} \quad J_{-1/2}(x) = C_2\sqrt{\frac{1}{x}}\cos x.$$

The constants C_1 and C_2 are arbitrary, but to make these results compatible with the normalization used for a_0 when developing the series solution for $J_\nu(x)$ we compare these expressions with the asymptotic formula (70), from which we see it is necessary to set $C_1 = C_2 = \sqrt{(2/\pi)}$, to obtain

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x \quad \text{and} \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x. \tag{72}$$

Expressions for $J_{\pm n/2}(x)$ now follow by use of recurrence relation (66). Thus, for example, setting $\nu = 1/2$ in (66) gives

$$J_{3/2}(x) = \frac{1}{x}J_{1/2}(x) - J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right), \tag{73}$$

and, similarly, setting $\nu = -1/2$ gives

$$J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left(\sin x + \frac{\cos x}{x}\right). \tag{74}$$

We have shown that all Bessel functions $J_{\pm n/2}(x)$ with n an odd integer are expressible in terms of elementary functions. The derivation of $J_{\pm 1/2}(x)$ directly from series (54) forms an exercise in the set at the end of this section.

FRIEDRICH WILHELM BESSEL (1784–1846)
A German mathematician who started his career as a clerk apprenticed to a mercantile office in Bremen where he remained for a number of years. Using published observations he calculated the orbit of Haley's comet and submitted his calculations to the astronomer H.W.M. Olbers who recognized his ability and, after recommending the work for publication, arranged for Bessel to become an assistant in the observatory in Lilienthal. His major mathematical contribution was the introduction, in a paper of 1824 devoted to planetary motions, of the class of transcendental functions now known as Bessel functions.

Summary

Bessel's equation was introduced and series solutions were obtained by the Frobenius method for the Bessel function $J_\nu(x)$ of the first kind of order ν. It was shown that Bessel functions of the first kind of fractional order $\pm n/2$, with n odd, could be expressed in terms of products of sines and cosines and $1/\sqrt{x}$.

EXERCISES 8.6

1. Write down the first six terms of the series expansion for $J_2(x)$.
2. Write down the first six terms of the series expansion for $J_3(x)$.
3. Derive result (65) by differentiating the product of $x^{-1/2}$ and the series for $J_\nu(x)$ given in (54).
4. Determine how many terms must be retained in the series for $J_0(x)$ for it to be accurate to four decimal places over the interval $0 \le x \le 4$.
5. Determine how many terms must be retained in the series for $J_0(x)$ for it to be accurate to four decimal places over the interval $0 \le x \le 2$.
6. Determine how many terms must be retained in the series for $J_0(x)$ for it to be accurate to six decimal places over the interval $0 \le x \le 1$.
7. Determine how many terms must be retained in the series for $J_0(x)$ for it to be accurate to six decimal places over the interval $0 \le x \le 2$.
8. Determine how many terms must be retained in the series for $J_1(x)$ for it to be accurate to four decimal places over the interval $0 \le x \le 2$.
9. Determine how many terms must be retained in the series for $J_1(x)$ for it to be accurate to four decimal places over the interval $0 \le x \le 3$.
10. Integrate the first four terms in the series for $J_0(x)$ term by term to obtain an approximation to

$$\int_0^x J_0(t)dt.$$

Estimate the maximum magnitude of the error when using the result in the interval $0 \le x \le a$.

11. Integrate the first four terms in the series for $J_1(x)$ term by term to obtain an approximation to

$$\int_0^x J_1(t)dt.$$

Estimate the maximum magnitude of the error when using the approximation in the interval $0 \le x \le a$. Integrate the integral analytically, and confirm that the analytical result and the approximation are in agreement.

The Bessel function $J_\nu(\lambda x)$ is a solution of $x^2y'' + xy' + (\lambda^2x^2 - \nu^2)y = 0$. Establish the following results by making the change of variable $x = \lambda X$ in results (64) to (67), and then replacing X by x.

12. $\dfrac{d}{dx}[x^\nu J_\nu(\lambda x)] = \lambda x^\nu J_{\nu-1}(\lambda x)$.
13. $\dfrac{d}{dx}[x^{-\nu} J_\nu(x)] = -\lambda x^{-\nu} J_{\nu+1}(\lambda x)$.
14. $\dfrac{d}{dx}[J_\nu(\lambda x)] = \lambda J_{\nu-1}(\lambda x) - \dfrac{\nu}{x} J_\nu(\lambda x)$.
15. $\dfrac{d}{dx}[J_\nu(\lambda x)] = -\lambda J_{\nu+1}(\lambda x) + \dfrac{\nu}{x} J_\nu(\lambda x)$.
16. $\dfrac{d}{dx}[J_\nu(\lambda x)] = \dfrac{\lambda}{2}[J_{\nu-1}(\lambda x) - J_{\nu+1}(\lambda x)]$.
17. $J_\nu(\lambda x) = \dfrac{\lambda x}{2\nu}[J_{\nu-1}(\lambda x) + J_{\nu+1}(\lambda x)]$.
18. Use $(64)'$ and $(65)'$ to show that

$$\frac{d}{dx}[xJ_\nu(x)J_{\nu+1}(x)] = x\left[J_\nu^2(x) - J_{\nu+1}^2(x)\right].$$

19. Show that $\lim_{x\to 0} J_0(x) = 1$, $\lim_{x\to 0} J_n(x) = 0$ for $n = 1, 2, \ldots$ and, $\lim_{x\to\infty} J_n(x) = 0$ for $n = 0, 1, \ldots,$ and prove that

$$\int_0^\infty J_1(x)dx = 1.$$

20. Use the results in Exercise 19 with (67) to show that

$$1 = \int_0^\infty J_1(x)ds = \int_0^\infty J_3(x)dx = \cdots$$
$$= \int_0^\infty J_{2n+1}(x)dx = \cdots \quad \text{for } n = 0, 1, \ldots.$$

21. In Section 7.3(d)(ii) it was shown that the Laplace transform of $J_0(x)$ was

$$\mathcal{L}\{J_0(x)\} = \frac{1}{(s^2+1)^{1/2}}.$$

Use this result to deduce the value of $\int_0^\infty J_0(x)dx$, and then use (67) together with the results of Exercise 20 to show that

$$1 = \int_0^\infty J_0(x)dx = \int_0^\infty J_1(x)dx = \int_0^\infty J_3(x)dx = \cdots$$
$$= \int_0^\infty J_n(x)dx = \cdots \quad \text{for } n = 0, 1, 2, \ldots.$$

22. Find (a) $\int x^3 J_2(x)\mathrm{d}x$ and (b) $\int x^{-3} J_4(x)\mathrm{d}x$.
23. Express $\int J_4(x)\mathrm{d}x$ in terms of $\int J_0(x)\mathrm{d}x$.

24. Express $\int J_5(x)\mathrm{d}x$ in terms of $J_0(x)$, $J_2(x)$, and $J_4(x)$.

25. Express $\int xJ_1(x)\mathrm{d}x$ in terms of $\int J_0(x)\mathrm{d}x$.

26. Express $\int x^2 J_0(x)\mathrm{d}x$ in terms of $\int J_0(x)\mathrm{d}x$.

The exercises that follow, some of which are slightly harder, provide background information about Bessel functions.

27.* By differentiating under the integral sign with respect to x, integrating by parts, and combining results using an elementary trigonometric identity, prove that

$$J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)d\theta$$

is an integral representation of $J_0(x)$ by showing that it satisfies Bessel's equation of order zero

$$xJ_0'' + J_0' + xJ_0 = 0.$$

28.* The function $\exp[\frac{x}{2}(t - \frac{1}{t})]$ is the **generating function** for the Bessel functions $J_n(x)$, and it has the property that when it is expanded in powers of t (both positive and negative),

$$\exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(x)t^n.$$

Thus, $J_n(x)$ is the coefficient of t^n in the expansion of the generating function in powers of t. Expand the exponential as the product of the series for $\exp[xt/2]$ and $\exp[-x/(2t)]$, and hence derive the first three terms of the series expansion of $J_0(x)$.

29.* Differentiate the generating function partially with respect to x and equate the coefficients of t^n on each side of the identity to prove that

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x).$$

30.* Differentiate the generating function partially with respect to t and equate the coefficients of t^{n-1} on each side of the identity to prove that

$$\frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x).$$

31.* Substitute $\nu = 1/2$ in (54) and (58), and hence show that $J_{1/2} = \sqrt{\frac{2}{\pi x}}\sin x$ and $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$.

32.* Use (66) together with results (73) and (74) to show that

$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3}{x^2} - 1\right)\sin x - \frac{3}{x}\cos x\right]$$

$$J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3}{x}\sin x + \left(\frac{3}{x^2} - 1\right)\cos x\right]$$

$$J_{9/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\left(\frac{105}{x^4} - \frac{45}{x} + 1\right)\sin x - \left(\frac{105}{x^3} - \frac{10}{x}\right)\cos x\right]$$

and

$$J_{-9/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\left(\frac{105}{x^3} - \frac{10}{x}\right)\sin x + \left(\frac{105}{x^4} - \frac{45}{x^2} + 1\right)\cos x\right].$$

8.7 Bessel Functions of the Second Kind $Y_\nu(x)$

It was shown in the previous section that, with the exception of $\nu = 1/2$, the two Bessel functions $J_\nu(x)$ and $J_{-\nu}(x)$ of the first kind are only linearly independent solutions of Bessel's equation when the roots of the indicial equation differ by an integer. So it remains for us to find a second linearly independent solution when $\nu = n$ and $n = 0, 1, 2, \ldots$. We begin by considering the case $n = 0$, corresponding to the repeated root $\nu = 0$, when it follows from Theorem 8.2(b) that the form of solution to be expected in the case of Bessel's equation of order zero

Bessel functions of the second kind

$$xy'' + y' + xy = 0 \tag{75}$$

is

$$y_2(x) = J_0(x)\ln x + \sum_{r=0}^{\infty} b_r x^{r+1}. \tag{76}$$

Differentiation of (76) gives

$$y_2'(x) = J_0'(x)\ln x + \frac{J_0(x)}{x} + \sum_{r=0}^{\infty}(r+1)b_r x^r$$

and

$$y_2''(x) = J_0''(x)\ln x + \frac{2J_0'(x)}{x} - \frac{J_0(x)}{x^2} + \sum_{r=0}^{\infty}(r+1)rb_r x^{r-1}.$$

When these expressions are substituted into (75) the terms in $J_0(x)$ cancel, causing the equation to reduce to

$$[xJ_0''(x) + J_0'(x) + xJ_0(x)]\ln x + 2J_0'(x) + \sum_{r=0}^{\infty}(r+1)rb_r x^r$$
$$+ \sum_{r=0}^{\infty}(r+1)b_r x^r + \sum_{r=0}^{\infty} b_r x^{r+2} = 0.$$

The logarithmic term vanishes because $J_0(x)$ is a solution of (75), so the coefficients b_r are determined by the equation

$$2J_0'(x) + \sum_{r=0}^{\infty}(r+1)rb_r x^r + \sum_{r=0}^{\infty}(r+1)b_r x^r + \sum_{n=0}^{\infty} b_r x^{r+2} = 0.$$

To proceed further it is necessary to determine $J_0'(x)$, but this can be found by differentiating (56) in Section 8.6. After cancellation of a factor $2m$ from the numerator and denominator of the resulting expression, and noticing that the summation now starts from $m = 1$, it is found that

$$J_0'(x) = \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-1}(m-1)!m!}.$$

Combining this with the previous result gives

$$\sum_{m=1}^{\infty} \frac{(-1)^{m+1} x^{2m+1}}{2^{2m}(m+1)!m!} + \sum_{r=0}^{\infty}(r+1)rb_r x^r + \sum_{r=0}^{\infty}(r+1)b_r x^r + \sum_{r=0}^{\infty} b_r x^{r+2} = 0.$$

Shifting the summation index in the last term and combining the summations reduces this to

$$\sum_{m=1}^{\infty} \frac{(-1)^{m+1} x^{2m+1}}{2^{2m}(m+1)!m!} + b_0 + 4b_1 x + \sum_{r=2}^{\infty}\{(r+1)^2 b_r + b_{r-2}\}x^r = 0.$$

We now make use of the fact that terms may be rearranged in an absolutely convergent series in order to rewrite the last summation as a sum of even powers of x and a sum of odd powers of x before combining the results. The preceding equation then becomes

$$\sum_{m=1}^{\infty} \frac{(-1)^{m+1} x^{2m+1}}{2^{2m}(m+1)!m!} + b_0 + 4b_1 x + \sum_{m=1}^{\infty}\{(2m+1)^2 b_{2m} + b_{2m-2}\}x^{2m}$$
$$+ \sum_{m=2}^{\infty}[4m^2 b_{2m-1} + b_{2m-3}]x^{2m-1} = 0.$$

Next we equate the coefficient of each power of x to zero in the usual manner. As there is no constant term in the first summation, it follows that $b_0 = 0$. The recurrence relation in the second summation is $(2m+1)^2 b_{2m} + b_{2m-2} = 0$, so together with the result $b_0 = 0$ this implies that $b_{2m} = 0$ for $m = 0, 1, 2, \ldots$. Setting the summation involving even powers of x to zero brings the equation into the form

$$\sum_{m=1}^{\infty} \frac{(-1)^{m+1}\, x^{2m+1}}{2^{2m}(m+1)!m!} + 4b_1 x + \sum_{m=2}^{\infty} \left[4m^2 b_{2m-1} + b_{2m-3}\right] x^{2m-1} = 0.$$

We now equate to zero the coefficients of each remaining power of x, and proceeding in this manner it is not difficult to show that the general coefficient b_{2m-1} can be written

$$b_{2m-1} = \frac{(-1)^{m-1}}{2^{2m}(m!)^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right), \quad \text{for } m = 1, 2, \ldots,$$

so the second linearly independent solution is

$$y_2(x) = J_0(x) \ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\, x^{2m}}{2^{2m}(m!)^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right). \tag{77}$$

Defining h_m as

$$h_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m} \tag{78}$$

allows $y_2(x)$ to be written in the more convenient form

$$y_2(x) = J_0(x) \ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\, h_m x^{2m}}{2^{2m}(m!)^2}. \tag{79}$$

The series in (79) can be shown to converge, though as the logarithmic term becomes infinite at the origin, result (79) is only finite for $x > 0$.

As any linear combination of two linearly independent solutions of a differential equation is itself a solution, it proves to be convenient to take as the second solution of Bessel's equation of order zero the function $Y_0(x)$ defined as the linear combination

$$Y_0(x) = \frac{2}{\pi}[y_2(x) + (\gamma - \ln 2) J_0(x)], \tag{80}$$

where the constant γ, called the **Euler constant**, is defined as

$$\gamma = \lim_{m\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m} - \ln m\right), \tag{81}$$

where $\gamma = 0.577\ 215\ 664\ 901\ldots$. This constant is also called the **Euler–Mascheroni constant**, and on occasion it is denoted by C and sometimes by $\ln \gamma$.

the Bessel functions $Y_0(x)$ and $Y_\nu(x)$

The function $Y_0(x)$, called the **Bessel function of the second kind of order zero**, is defined as

$$Y_0(x) = \frac{2}{\pi}\left[J_0(x)\left(\ln\frac{x}{2}+\gamma\right)+\sum_{m=1}^{\infty}\frac{(-1)^{m-1}h_m}{2^{2m}(m!)^2}x^{2m}\right]. \tag{82}$$

The reason for choosing this particular combination of functions in the definition of $Y_0(x)$ is because of its convenient properties as $x \to \infty$. The function $Y_0(x)$ is also called the **Neumann** or **Weber function** of order zero and denoted by $N_0(x)$.

Some authors make a distinction in what they call a Bessel function of the second kind, so there may be a difference between the Weber function $Y_n(x)$ and the Neumann function $N_n(x)$. Because of this, care must be exercised when using these functions in software packages.

Bessel functions of the second kind of integral order can be defined in similar fashion, but to make them compatible with the functions $J_{-\nu}(x)$ introduced in Section 8.6 the following definition is adopted:

$$Y_\nu(x) = \frac{1}{\sin\nu\pi}[J_\nu(x)\cos\nu\pi - J_{-\nu}(x)] \tag{83}$$

with

$$Y_n(x) = \lim_{\nu\to n} Y_\nu(x). \tag{84}$$

Using this last result it is possible to show that for integral values of ν the function $Y_n(x)$ is given by

$$\begin{aligned} Y_n(x) = {} & \frac{2}{\pi}J_n(x)\left(\ln\frac{x}{2}+\gamma\right)+\frac{x^n}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m-1}(h_m+h_{m+n})}{2^{2m+n}\,m!(m+n)!}x^{2m} \\ & -\frac{1}{\pi x^n}\sum_{m=0}^{n-1}\frac{(n-m-1)!}{2^{2m-n}m!}x^{2m} \end{aligned} \tag{85}$$

where, by definition, $h_0 = 1$. It follows from this that the Bessel functions $Y_n(x)$ and $Y_{-n}(x)$ are linearly dependent, with

$$Y_{-n}(x) = (-1)^n Y_n(x).$$

Graphs of the first three Bessel functions of the second kind are shown in Fig. 8.7.

When x is small the following approximations are useful:

$$Y_0(x) \approx \frac{2}{\pi}\ln x \quad \text{and for } \nu > 0, \quad Y_\nu(x) \approx -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu. \tag{86}$$

asymptotic form for $Y_\nu(x)$

For large x, however, the asymptotic approximation to $Y_\nu(x)$ is

$$Y_\nu(x) \sim \sqrt{\frac{2}{\pi x}}\sin\left[x-\left(\frac{2\nu+1}{4}\right)\pi\right]. \tag{87}$$

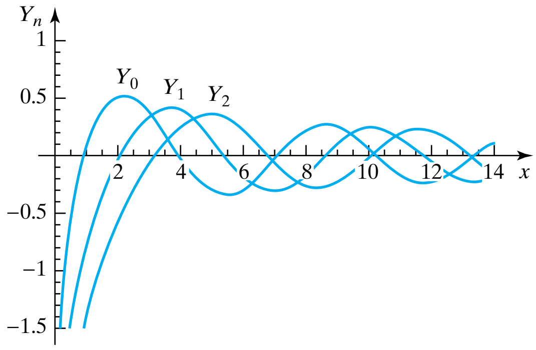


FIGURE 8.7 Bessel functions $Y_0(x)$, $Y_1(x)$, and $Y_2(x)$ of the second kind.

It follows from (86) and (87) that

$$\lim_{x\to 0} Y_\nu = -\infty \quad \text{and} \quad \lim_{x\to\infty} Y_\nu(x) = 0. \tag{88}$$

zeros of Bessel functions $Y_n(x)$

The zeros of $Y_n(x)$ are needed when working with Bessel functions, so the locations of the first six zeros of $Y_n(x)$ for $n = 0, 1, 2, 3$ are listed in Table 8.2. The rth zero of the Bessel function $Y_n(x)$ is denoted by $y_{n,r}$, so, for example, the second zero of $Y_1(x)$ is $y_{1,2} = 5.42968$.

It is a consequence of the definition of $Y_\nu(x)$ that for all ν the general solution of Bessel's equation in the standard form

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \tag{89}$$

is

$$y(x) = C_1J_\nu(x) + C_2Y_\nu(x). \tag{90}$$

Similarly, the general solution of Bessel's equation in the form

$$x^2y'' + xy' + (\lambda^2x^2 - \nu^2)y = 0 \tag{91}$$

is

$$y(x) = C_1J_\nu(\lambda x) + C_2Y_\nu(\lambda x). \tag{92}$$

TABLE 8.2 Zeros $y_{n,r}$ of $Y_n(x)$ for $n = 0, 1, 2, 3$

r	$y_{0,r}$	$y_{1,r}$	$y_{2,r}$	$y_{3,r}$
1	0.89358	2.19714	3.38424	4.52702
2	3.95786	5.42968	6.79381	8.09755
3	7.08605	8.59601	10.02348	11.39647
4	10.22235	11.74915	13.20999	14.62308
5	13.36110	14.89744	16.37897	17.81846
6	16.50092	18.04340	19.53904	20.99728

Many differential equations can be solved in terms of Bessel functions after a suitable transformation of the dependent variable. In particular, the equation

$$y'' + \left(\frac{1-2a}{x}\right) y' + \left[b^2c^2x^{2c-2} + \left(\frac{a^2 - \nu^2c^2}{x^2}\right)\right] y = 0 \tag{93}$$

can be shown to have the solution

$$y(x) = x^a Z_\nu(bx^c), \tag{94}$$

where a, b, and c are numbers and Z_ν is any linear combination of J_ν and Y_ν (see Exercise 16 at the end of this section).

The following is an application of Bessel functions to a simple physical problem. It illustrates how, in this case, the conditions of the problem only allow a Bessel function of the first kind to be retained in the solution. The problem, which is a classical one, can be stated as follows.

Find the radial temperature distribution $T(r)$ in a wire of circular cross-section with $0 \le r \le R$, when the electrical conductivity is σ, the thermal conductivity is K, and the wire carries a uniform current of density I amps per unit area of cross-section. Assume that the temperature at the center of the wire is T_0 and that the resistance of the wire varies linearly with the temperature as $\alpha T(r)$, with α a constant.

In order to formulate the problem in mathematical terms, we begin with the fact that the rate of heat generation in a unit volume of the wire is given by JI^2/σ heat units, where J is a physical constant (typically the number of calories in a joule). It follows from arguments given later in Chapter 18 that the equation determining the radial steady state temperature distribution is

$$K\frac{d^2T}{dr^2} + \frac{K}{r}\frac{dT}{dr} + \frac{\alpha JI^2}{\sigma}T = -\frac{JI^2}{\sigma},$$

where the last term on the left takes account of the linear variation of resistance with temperature, and the term on the right represents the heat generation due to the current.

When divided by K, this is seen to be Bessel's equation of order zero with a nonhomogeneous term $-JI^2/K\sigma$, and it is easily shown to have the general solution

$$T(r) = AJ_0\left(Ir\sqrt{\frac{\alpha J}{K\sigma}}\right) + BY_0\left(Ir\sqrt{\frac{\alpha J}{K\sigma}}\right) - \frac{1}{\alpha},$$

with A and B arbitrary constants. As the temperature must remain finite at the center of the wire, we must set $B = 0$ to remove the infinite value of Y_0 when $r = 0$. However, $T(0) = T_0$, so $A = T_0 + 1/\alpha$ and the required radial temperature distribution becomes

$$T(r) = \left(T_0 + \frac{1}{\alpha}\right) J_0\left(Ir\sqrt{\frac{\alpha J}{K\sigma}}\right) - \frac{1}{\alpha} \quad \text{for} \quad 0 \le r \le R. \qquad \blacksquare$$

Summary

It was seen in the previous section that when n is an integer $J_n(x)$ and $J_{-n}(x)$ are linearly dependent. This section has shown how a second linearly independent solution $Y_\nu(x)$ can be constructed that for all ν is linearly independent of $J_\nu(x)$, so the general solution of Bessel's equation can always be written $y(x) = AJ_\nu(x) + BY_\nu(x)$, where A and B are arbitrary constants. The function $Y_\nu(x)$ is called a Bessel function of the second kind of order ν.

EXERCISES 8.7

In Exercises 1 through 10, find the general solution of the differential equation.

1. $x^2y'' + xy' + (x^2 - 4)y = 0.$
2. $4x^2y'' + 4xy' + (4x^2 - 1)y = 0.$
3. $xy'' + y' + xy = 0.$
4. $xy'' + y' + \lambda^2 xy = 0.$
5. $xy'' + y' + 4x^3y = 0$; substitute $u = x^2$.
6. $x^2y'' + 3xy' + (x^2 + 1)y = 0$; substitute $y = u/x$.
7. $x^2y'' + xy' + 4(x^2 - 1)y = 0.$
8. $xy'' + y' + 9x^5y = 0$; substitute $u = x^3$.
9. $4x^2y'' + (16x^2 + 1)y = 0$; substitute $y = x^{1/2}u$.
10. $x^2y'' + 5xy' + (x^2 + 4)y = 0$; substitute $y = u/x^2$.

Use (93) and (94) to find the solution of the differential equations in Exercises 11 through 15.

11. $x^2y'' - xy' + (4x^4 - 3)y = 0.$
12. $xy'' - 3y' + xy = 0.$
13. $x^2y'' - xy' + (9x^2 + 1)y = 0.$
14. $x^2y'' - 5xy' + (16x^4 + 1)y = 0.$
15. $x^2y'' - 3xy' + (64x^8 - 8)y = 0.$
16. Verify that $y(x) = x^a Z_\nu(bx^c)$ is a solution of (93) by substituting for $y(x)$ in the differential equation and showing that this leads to the equation
$$X^2 Z_\nu''(X) + XZ'_\nu(X) + (X^2 - \nu^2)Z_\nu(X) = 0,$$
with $X = bx^c$. Hence, conclude that $Z_\nu(X)$ is either $J_\nu(X)$ or $Y_\nu(X)$, and so, because of the linearity of the equation, $Z_\nu(X) = C_1J_\nu(X) + C_2Y_\nu(X)$ must be a solution.
17. Use the substitution $y(x) = x^{-\nu}u(x)$ to convert the equation
$$x^2\frac{d^2y}{dt^2} + ax\frac{dy}{dx} + (1 + k^2x^2)y = 0,$$
in which a is a parameter, into an equation for $u(x)$. Find the values of a and ν that make the equation in $u(x)$ Bessel's equation of order zero. Use the result to find the general solution $y(x)$ that corresponds to this value of a.

8.8 Modified Bessel Functions $I_\nu(x)$ and $K_\nu(x)$

Replacing the independent variable x in Bessel's equation by ix changes the differential equation to

$$x^2y'' + xy' - (x^2 + \nu^2)y = 0, \tag{95}$$

Bessel's modified equation

called **Bessel's modified equation of order** ν.

It follows directly from Section 8.7 that Bessel's modified equation has two linearly independent complex solutions $J_\nu(ix)$ and $Y_\nu(ix)$. These solutions are not convenient to use, so the process of scaling and combining linearly independent solutions of a linear differential equation to form other solutions is used to produce two real linearly independent solutions denoted by $I_\nu(x)$ and $K_\nu(x)$. These are called, respectively, **modified Bessel functions of the first and second kinds of order** ν.

The modification of $J_\nu(ix)$ is straightforward, because from (54)

$$J_\nu(ix) = \sum_{m=0}^{\infty}\frac{(-1)^m(ix)^{2m+\nu}}{2^{2m+\nu}m!\Gamma(m+1+\nu)} = i^\nu\sum_{m=0}^{\infty}\frac{x^{2m+\nu}}{2^{2m+\nu}m!\Gamma(m+1+\nu)},$$

so the factor i^ν is removed and the **modified Bessel function of the first kind of order** ν is defined as the real function

$$I_\nu(x) = \sum_{m=0}^{\infty}\frac{x^{2m+\nu}}{2^{2m+\nu}\, m!\Gamma(m+1+\nu)}. \tag{96}$$

Unlike the series for $J_\nu(x)$, the series for $I_\nu(x)$ in (96) is no longer an alternating series, though it converges rapidly. As with ordinary Bessel functions, provided ν is not an integer, the general solution of Bessel's modified equation (95) can be written

the modified Bessel functions $I_\nu(x)$ and $K_\nu(x)$

$$y(x) = C_1 I_\nu(x) + C_2 I_{-\nu}(x). \tag{97}$$

However, rather than use $I_{-\nu}(x)$, in its place it is usual to introduce the real function $K_\nu(x)$ defined as the linear combination of real functions

$$K_\nu(x) = \left(\frac{\pi}{2}\right)\left(\frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi}\right), \tag{98}$$

and to call $K_\nu(x)$ the **modified Bessel function of the second kind of order** ν. It can be seen from (98) that the functions $I_\nu(x)$ and $K_\nu(x)$ are linearly independent.

The definition of $K_\nu(x)$ can be extended to the case in which ν is an integer n by defining the function $K_n(x)$ as

$$K_n(x) = \lim_{\nu \to n}\left(\frac{\pi}{2}\right)\left(\frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi}\right). \tag{99}$$

Because of this extension of the definition of $K_\nu(x)$, the general solution of Bessel's modified equation (95) can always be written in the form

$$y(x) = C_1 I_\nu(x) + C_2 K_\nu(x), \tag{100}$$

with no restriction placed on ν. The function $K_\nu(x)$ is also sometimes called the Kelvin function.

Similarly, when Bessel's modified equation is written in the form

$$x^2 y'' + xy' - (\lambda^2 x^2 + \nu^2)y = 0, \tag{101}$$

its general solution is given by

$$y(x) = C_1 I_\nu(\lambda x) + C_2 K_\nu(\lambda x), \tag{102}$$

with no restriction placed on ν.

This definition of $K_0(x)$ leads to the expansion

$$\begin{aligned} K_0(x) = -\left[\ln \frac{x}{2} + \gamma\right] I_0(x) + \frac{x^2/4}{(1!)^2} + \left(1 + \frac{1}{2}\right)\frac{(x^2/4)^2}{(2!)^2} \\ + \left(1 + \frac{1}{2} + \frac{1}{3}\right)\frac{(x^2/4)^3}{(3!)^2} + \cdots, \end{aligned} \tag{103}$$

with similar though more complicated expansions for $K_n(x)$.

Graphs of $I_0(x)$ and $I_1(x)$ and of $K_0(x)$ and $K_1(x)$ are shown in Figs. 8.8 and 8.9, respectively.

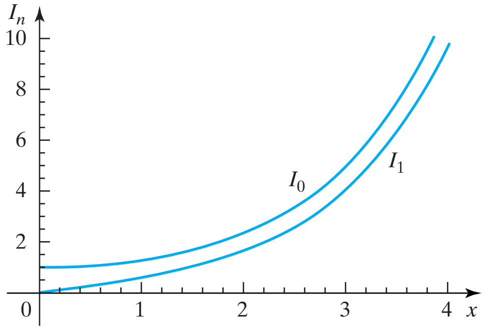


FIGURE 8.8 Graphs of $I_0(x)$ and $I_1(x)$.

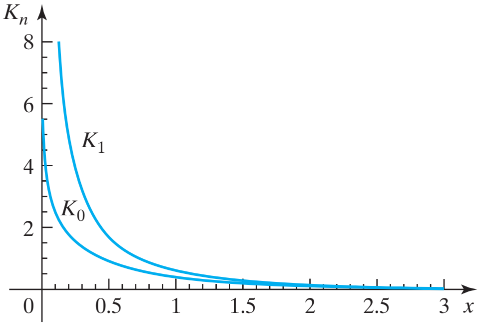


FIGURE 8.9 Graphs of $K_0(x)$ and $K_1(x)$.

The following are useful properties of $I_\nu(x)$ and $K_\nu(x)$:

$$\begin{aligned} &I_0(0) = 1, \quad I_n(0) = 0 \quad \text{for } n = 1, 2, \ldots, \lim_{x\to 0} I_\nu(x) = 0, \\ &K_n(0) = \infty, \quad \lim_{x\to\infty} K_n(x) = 0 \quad \text{for } n = 0, 1, 2, \ldots. \end{aligned} \tag{104}$$

asymptotic expressions for modified Bessel functions

For small x

$$\begin{aligned} &I_\nu(x) \sim \frac{1}{\Gamma(1+\nu)}\left(\frac{x}{2}\right)^\nu, \quad K_0(x) = -\ln x \quad \text{and} \\ &\qquad K_\nu(x) \sim \frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^\nu \quad \text{for } \nu > 0, \end{aligned} \tag{105}$$

whereas for large x

$$I_\nu(x) \approx \frac{1}{\sqrt{2\pi x}}e^x \quad \text{and} \quad K_\nu(x) \approx \sqrt{\frac{\pi}{2x}}e^{-x}. \tag{106}$$

Results involving Bessel functions of the first and second kinds, together with applications, are to be found in Chapter 5 of reference [3.7]. Chapters 9 to 11 of Reference [G.1] and Chapter 17 of reference [G.3] give general information about all types of Bessel functions. The standard encyclopedic work covering all aspects of Bessel functions is reference [3.17].

Summary

Modified Bessel functions were introduced, their series solutions were obtained, the general solution was expressed in terms of $I_\nu(x)$ and $K_\nu(x)$, and asymptotic representations were given.

EXERCISES 8.8

1. By differentiating the series for $I_0(x)$, show that $I_0'(x) = I_1(x)$.

2. Use the definition of $I_\nu(x)$ to show that

$$I_{\nu-1}(x) - I_{\nu+1}(x) = \frac{2\nu}{x}I_\nu(x) \quad \text{for } \nu \geq 1.$$

3. Use the definition of $I_\nu(x)$ to show that

$$I_{\nu-1}(x) + I_{\nu+1}(x) = 2I_\nu'(x) \quad \text{for } \nu \geq 1.$$

4. Use Lemma 6.1 of Section 6.3 to reduce Bessel's modified equation of order $\nu = 1/2$ to standard form, and

hence show that

$$I_{1/2}(x) \text{ is proportional to } \frac{\sinh x}{\sqrt{x}}, \quad \text{and}$$
$$I_{-1/2}(x) \text{ is proportional to } \frac{\cosh x}{\sqrt{x}}.$$

5. Use asymptotic result (106) for $I_\nu(x)$ when x is large to find the constants of proportionality in Exercise 4, and then use the result of Exercise 2 to find $I_{3/2}(x)$ and $I_{-3/2}(x)$.

6. Use Lemma 6.1 of Section 6.3 to reduce Bessel's modified equation of order $\nu = 1/2$ to standard form, and hence show that when x is large two linearly independent solutions of the equation are proportional to $e^x/\sqrt{x}$ and $e^{-x}/\sqrt{x}$.

7. Deduce the expressions for $I_{\pm 1/2}(x)$ and $I_{\pm 3/2}(x)$ from the corresponding results for $J_{\pm 1/2}(x)$ and $J_{\pm 3/2}(x)$ in (72) to (74) of Section 8.6.

8. Use Abel's formula in Exercise 6 of set 6.1 to show that if y_1 and y_2 are any two linearly independent solutions of Bessel's modified equation, then

$$y_1 y_2' - y_2 y_1' = C/x,$$

where C is a constant introduced through the Abel formula.

9. Set $y_1(x) = I_\nu(x)$ and $y_2(x) = I_{-\nu}(x)$ in the result of Exercise 8, where ν is not an integer. Substitute the series for $I_\nu(x)$ and $I_{-\nu}(x)$, and by finding the coefficient of $1/x$ on the left-hand side identify the coefficient C. Use the result

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

to show that

$$I_\nu(x) I_{-\nu}'(x) - I_\nu'(x) I_{-\nu}(x) = -\frac{2}{\pi x} \sin \nu x.$$

10. Use the definition of $K_\nu(x)$ with the result of Exercise 9 to show that

$$I_\nu(x) K_\nu'(x) - I_\nu'(x) K_\nu(x) = -\frac{1}{x}.$$

11.* The amplitude $R(r)$ of the small symmetric vibrations of a flexible annular disc $a \le r \le b$ normal to its surface with its outer edge free and its inner edge fixed to a rod that oscillates along its length is governed by the equation

$$\frac{d^4 R}{dr^4} + \frac{2}{r}\frac{d^3 R}{dr^3} - \frac{1}{r^2}\frac{d^2 R}{dr^2} + \frac{1}{r^3}\frac{dR}{dr} - R = 0.$$

Show by expressing the equation as

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - 1\right)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + 1\right) R = 0$$

that its general solution is

$$R(r) = AJ_0(r) + BY_0(r) + CI_0(r) + DK_0(r),$$

where A, B, C, and D are arbitrary constants.

12.* In partial differential equations that govern physical phenomena with cylindrical and spherical polar coordinates, the following equation describes the radial variation $R(r)$ of the solution as a function of the radius r (see Chapter 18):

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\lambda^2 - \frac{n^2}{r^2}\right) R = 0.$$

Here, λ is a parameter and $n = 0, 1, 2, \ldots$. Show that the general solution of the equation is

$$R(r) = AJ_n(\lambda r) + BY_n(\lambda r).$$

Find the form of the solution of the following boundary value problems, given that $R(r)$ remains bounded, and determine the permissible values of the parameter λ.

(i) $0 \le r \le a$, for all n with the boundary conditions $R(a) = 0$.

(ii) $b \le r \le c$, for all n with the boundary conditions $R(b) = R(c) = 0$.

(iii) $0 \le r \le a$, for all n with the boundary conditions $R(a) + kR'(a) = 0 (k = \text{const})$.

(iv) $b \le r \le c$, for $n = 0$ with the boundary conditions $R(b) = R'(c) = 0$.

8.9 A Critical Bending Problem: Is There a Tallest Flagpole?

The implication of the question posed in the section heading will have been experienced by anyone who has tried holding a long, thin, flexible rod in a vertical position. If the rod is short, and its tip is given a small sideways displacement and released, the rod will perform transverse oscillations until it reaches an equilibrium position in a bent shape because of supporting its own weight. The longer the rod, the larger the amplitude of these oscillations, and the greater the bending under its

own weight when in equilibrium, until at some critical length the rod will bend until its tip just touches the ground, after which it will remain in that position.

Bessel functions and the bending of a thin vertical rod

An idealization of this phenomenon can be modeled by a long, thin, flexible flagpole of uniform cross-section, the base of which is clamped in the ground so the pole is vertical. We then ask at what length will the pole become unstable, so that any displacement of the top of the pole will cause it to bend under its own weight until the top of the pole touches and remains in contact with the ground? This question can be posed in mathematical terms, and it is the one that will be answered here.

The solution to this question will involve the use of Bessel functions, but the linear differential equation involved will have to satisfy a two-point boundary condition instead of the initial conditions we have considered so far. This means that the existence and uniqueness of solutions to initial value problems guaranteed by Theorem 6.2 no longer applies, so even when a solution can be found it may not be unique — more will be said about this later.

Let us model the problem by considering a thin uniform flexible rod of length L with a constant cross-section that is constructed from material with a Young's modulus of elasticity E, with the moment of inertia of a cross-section about a diameter normal to the plane of bending equal to I. The line density along the rod will be assumed to be constant and equal to w. The x-axis will be taken to be vertical and to coincide with the undistorted axis of the rod, with its origin located at the base of the rod. The horizontal displacement of the rod at a position x will be taken to be y, as shown in Fig. 8.10.

It is known from Section 5.2(f) that if the moment acting on the rod at a position x is $M(x)$, the equation governing its transverse deflection y when in equilibrium is

$$EI\frac{d^2y}{dx^2} = M(x). \tag{107}$$

The **shear** on the rod at point x is the force exerted perpendicular to the axis of the rod at x due to the weight of the rod extending from x to the top at P. As the length of this part of the rod is $L - x$, and its line density is w, the weight of this section is

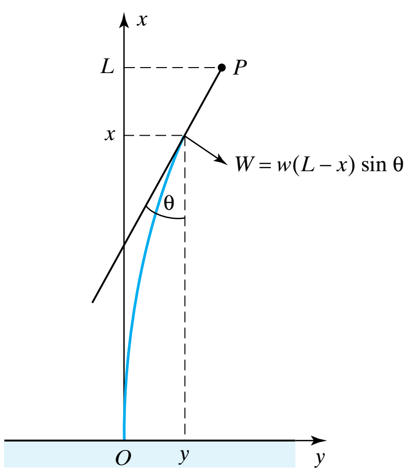


FIGURE 8.10 Equilibrium position of the rod when bent under its own weight.

given by $w(L-x)$, so the component W of this force normal to the axis of the rod at x is simply

$$W = w(L-x)\sin\theta, \tag{108}$$

where θ is the angle of deflection of the rod from the vertical at point x, as shown in Fig. 8.10.

It is known from mechanics that the shear on a rod is given in terms of the moment $M(x)$ by

$$\frac{dM}{dx} = -W(x). \tag{109}$$

We now make the approximation that the deflection at point x on the rod is small, so $\sin\theta \approx \tan\theta = dy/dx$, and by combining (107) to (109) we arrive that the governing equation for the deflection, which is the third order linear variable coefficient differential equation

$$EI\frac{d^3y}{dx^3} + w(L-x)\frac{dy}{dx} = 0. \tag{110}$$

Making the change of variable $z = L - x$ brings (110) to the more convenient form

$$\frac{d^3y}{dz^3} + \left(\frac{w}{EI}\right)z\frac{dy}{dz} = 0. \tag{111}$$

To apply this to our problem it is necessary to determine appropriate boundary conditions to be applied at the base and top of the rod. An obvious condition to be applied at the base is that due to clamping the pole in a vertical position at the origin, $(dy/dx)_{x=0} = (dy/dz)_{z=L} = 0$. To arrive at a second condition we notice that when the rod is bent and in equilibrium, there can be no bending moment at the top of the rod, so it can have no curvature at that point. Recalling that the radius of curvature ρ of a plane curve $y = y(x)$ is

$$\rho = \frac{(1+(y')^2)^{2/3}}{y''}, \tag{112}$$

we see that the rod will have no curvature at $x = L$ (equivalently at $z = 0$) when $\rho = \infty$, corresponding to $(d^2y/dx^2)_{x=L} = (d^2y/dz^2)_{z=0} = 0$.

Setting $u(z) = dy/dz$, these two boundary conditions become

$$u(L) = 0 \quad \text{and} \quad (du/dz)_{z=0} = 0. \tag{113}$$

Equation (111) is third order, but in terms of $u(z)$ it is only second order, and we have found two conditions on $u(z)$ from which to determine u. Fortunately, we only need to work with $u(z)$ to solve our problem. This is because we will soon see that the two-point boundary conditions (113) applied to the differential equation for u

$$\frac{d^2u}{dz^2} + \left(\frac{w}{EI}\right)zu = 0 \tag{114}$$

will provide sufficient information for us to find the critical length at which bending occurs.

Identifying equation (114) with (93) from Section 8.7, with x replaced by z, shows that

$$1-2a = 0, \quad 2c-2 = 1, \quad a^2 - \nu^2c^2 = 0, \quad \text{and} \quad b^2c^2 = w/EI, \tag{115}$$

so

$$a = 1/2, \quad c = 3/2, \quad \nu = 1/3, \quad \text{and} \quad b = \frac{2}{3}\sqrt{\frac{w}{EI}}. \tag{116}$$

Using this information in the solution (94) to equation (93) in Section 8.7 gives

$$u(z) = C_1\sqrt{z}\,J_{1/3}\left(\frac{2}{3}\sqrt{\frac{w}{EI}}z^{3/2}\right) + C_2\sqrt{z}\,J_{-1/3}\left(\frac{2}{3}\sqrt{\frac{w}{EI}}z^{3/2}\right). \tag{117}$$

Noticing from (71) of Section 8.6 that for small z

$$J_\nu(z) \approx \frac{1}{\Gamma(1+\nu)}\left(\frac{z}{2}\right)^\nu \quad \text{and} \quad J_{-\nu}(z) \approx \frac{1}{\Gamma(1-\nu)}\left(\frac{z}{2}\right)^{-\nu},$$

we see that close to the top of the rod, that is, for small z, $u(z)$ can be approximated by

$$u(z) \approx C_1\frac{z}{\Gamma(4/3)}\left(\frac{1}{3}\sqrt{\frac{w}{EI}}\right)^{1/3} + C_2\frac{1}{\Gamma(2/3)}\left(\frac{1}{3}\sqrt{\frac{w}{EI}}\right)^{-1/3}.$$

Differentiation of this result gives

$$u'(z) \approx C_1\frac{1}{\Gamma(4/3)}\left(\frac{1}{3}\sqrt{\frac{w}{EI}}\right)^{1/3},$$

but to satisfy the second boundary condition $(du/dz)_{z=0} = 0$, we must set $C_1 = 0$, causing solution (117) to reduce to

$$u(z) = C_2\sqrt{z}\,J_{-1/3}\left(\frac{2}{3}\sqrt{\frac{w}{EI}}z^{3/2}\right). \tag{118}$$

Applying the remaining boundary condition $u(L) = 0$ to (118) gives

$$0 = C_2\sqrt{L}\,J_{-1/3}\left(\frac{2}{3}\sqrt{\frac{w}{EI}}L^{3/2}\right), \tag{119}$$

and this will be satisfied if either $C_2 = 0$ or $J_{-1/3}(\frac{2}{3}\sqrt{\frac{w}{EI}}L^{3/2}) = 0$. The first condition $C_2 = 0$ corresponds to the unstable equilibrium configuration in which the rod is vertical, and so must be rejected, whereas the second condition corresponds to the required critical bending condition, and it will be satisfied when L is such that it causes $J_{-1/3}$ to vanish.

It is at this stage that we discover the boundary value problem does *not* have a unique solution, because the asymptotic behavior of $J_{-1/3}$ given in (70) of Section 8.6 shows that it has infinitely many zeros. To resolve this difficulty, and to find the length at which critical bending occurs, we must now seek a selection criterion for the length from *outside* the description of the physical situation provided by the differential equation.

Such a criterion is not hard to find, because critical bending must occur at the *smallest* value of L, say at L_c, that satisfies the condition

$$J_{-1/3}\left(\frac{2}{3}\sqrt{\frac{w}{EI}}L_c^{3/2}\right) = 0, \tag{120}$$

because if critical bending occurs when $L = L_c$, it will certainly occur at any larger value of L.

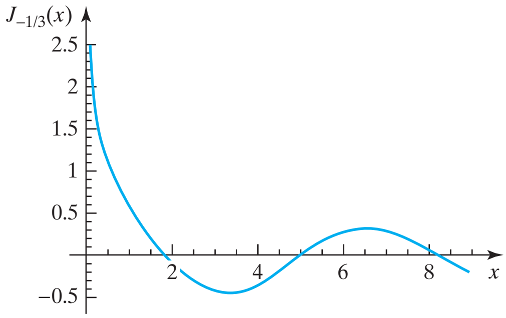


FIGURE 8.11 Graph of $J_{-1/3}(x)$ showing its first few zeros.

A graph of $J_{-1/3}(x)$ is shown in Fig. 8.11, from which it can be seen that the first zero α of $J_{-1/3}(x)$ occurs at around the value $\alpha \approx 1.87$, though numerical calculation provides the more accurate value $\alpha = 1.86635\ldots$. However, this accuracy is unnecessary, because the approximations made when modeling the physical situation introduce errors of sufficient magnitude that the value $\alpha \approx 1.87$ is adequate.

Using the value $\alpha = 1.87$ shows that the length L_c for critical bending must satisfy the formula

$$\frac{2}{3}\sqrt{\frac{w}{EI}}L_c^{3/2} \approx 1.87,$$

which is equivalent to

$$L_c \approx 1.99\left(\frac{EI}{w}\right)^{1/3}.$$

This approximation shows, as would be expected, that if the rod is not cylindrically symmetric about its axis, the critical length L_c will depend on the plane in which bending occurs, because the moment of inertia will depend on the direction in which the rod bends. Thus, for example, the critical length of a rod with a rectangular cross-section that bends in a plane parallel to one pair of its faces will differ from the critical length when bending occurs in a plane parallel to its other pair of faces. In such cases the model used is too simple, because twisting (torsion) will be likely to occur, causing the rod always to buckle in such a way that L_c assumes its smallest possible value.

The simplest case arises when the rod has a circular cross-section of radius a, for then the moment of inertia of the cross-section about any diameter is $I = \pi a^4/4$. When this expression is substituted into the approximation for L_c, we obtain the approximation

$$L_c \approx 1.25\left(\frac{Ea^4}{w}\right)^{1/3}.$$

Summary

In addition to involving Bessel functions, this idealization of a physical problem has illustrated the way in which a mathematical approach can sometimes lead to more than one solution, only one of which can be regarded as an approximation to the situation in the real world. The choice of the appropriate solution was seen to be based on an additional physical consideration that was outside the original formulation of the mathematical

problem. This situation is not unusual in applied mathematics, where the choice of solution is often based on stability considerations, a physically possible solution being stable, whereas a nonphysical solution is unstable and so will not be observed. A different example occurs in the study of shock waves in air where two solutions are mathematically possible, though only one is physically realizable. In that case the selection principle is based on the thermodynamics of the problem, though it can also be based on stability considerations.

8.10 Sturm–Liouville Problems, Eigenfunctions, and Orthogonality

Mathematical models of physical situations arising in engineering and physics lead to two-point boundary value problems for a function $y(x)$ that is defined over an interval $a < x < b$ and satisfies a differential equation of the form

$$y''(x) + P(x)y'(x) + (Q(x) + \lambda R(x))y(x) = 0, \tag{121}$$

in which λ is a parameter. This equation always has the solution $y(x) \equiv 0$, called the **trivial solution**, but if it is to have nontrivial solutions (solutions that are not identically zero) satisfying boundary conditions at $x = a$ and $x = b$, the parameter λ cannot be arbitrary. In what follows our purpose will be to find constant values of λ for which nontrivial solutions exist satisfying given boundary conditions. It will be seen later how these nontrivial solutions can be used to generalize series expansions of arbitrary functions over the interval $a < x < b$ that, along with other uses, are needed in Chapter 18 when solving partial differential equations by the method of separation of variables.

To proceed further we will write (121) in a more convenient form, and to this end we simplify its first two terms using the method developed in Section 5.6 when finding an integrating factor for a linear first order equation. Defining the function $p(x)$ as

$$p(x) = \exp\left[\int P(x)\mathrm{d}x\right],$$

and multiplying (121) by $p(x)$ gives

$$p(x)[y''(x) + P(x)y'(x)] + p(x)(Q(x) + \lambda R(x))y(x) = 0.$$

However,

$$p(x)[y''(x) + P(x)y'(x)] = \frac{d}{dx}\left[p(x)\frac{dy(x)}{dx}\right],$$

so the equation becomes

$$\frac{d}{dx}\left[p(x)\frac{dy(x)}{dx}\right] + p(x)(Q(x) + \lambda R(x))y(x) = 0.$$

Finally, setting $q(x) = p(x)Q(x)$ and $r(x) = p(x)R(x)$ allows equation (121) to be written in the form

$$\frac{d}{dx}\left[p(x)\frac{dy(x)}{dx}\right] + [q(x) + \lambda r(x)]y(x) = 0. \tag{122}$$

In what follows $p(x), q(x), r(x)$, and $p'(x)$ will be assumed to be continuous functions defined on a closed interval $a \le x \le b$ on which $p(x) > 0, r(x) > 0$.

Differential equations with these properties and written in this form are called **Sturm–Liouville equations**, and the type of boundary conditions that are to be imposed will be introduced after the following typical examples of these equations.

JACQUES CHARLES FRANÇOIS STURM (1803–1855) AND JOSEPH LIOUVILLE (1809–1882)
Sturm, who was born in Geneva, Switzerland, was Poisson's successor in the Chair of Mechanics in the Sorbonne. Much of his work was in algebra, where he worked on the determination of intervals on the real line inside each of which was located one real root of a polynomial, though he also worked on the study of heat flow introduced by his contemporary Joseph Fourier. Liouville, a professor at the Collège de France, also studied algebraic problems and, in particular, quadratic forms, though he also made contributions to elliptic functions and to complex analysis. Sturm and Liouville, who were friends, collaborated on the eigenvalue and eigenfunction problems raised by the study of heat flow, and together their work led to what is now called the study of Sturm–Liouville systems.

examples of Sturm–Liouville equations

Simple harmonic motion equation

The differential equation describing undamped simple harmonic oscillations

$$y'' + n^2 y = 0 \tag{123}$$

follows from (122) by setting $p(x) = 1$, $q(x) = 0$, $r(x) = 1$, and $\lambda = n^2$.

The Legendre equation

The Legendre equation encountered in (10) of Section 8.2, usually written

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0, \tag{124}$$

follows from (122) by setting $p(x) = 1 - x^2$, $q(x) = 0$, $r(x) = 1$, and $\lambda = \alpha(\alpha + 1)$.

Bessel's equation

When Bessel's equation of order ν is written in its more general form

$$x^2 y'' + xy' + (k^2x^2 - \nu^2)y = 0, \tag{125}$$

the equation follows from (122) by setting $p(x) = x$, $q(x) = -\nu^2/x$, $r(x) = x$, and $\lambda = k^2$.

The Chebyshev equation

The Chebyshev equation of order ν is

$$(1 - x^2)y'' - xy' + n^2 y = 0, \tag{126}$$

and the equation follows from (122) by setting $p(x) = (1 - x^2)^{1/2}$, $q(x) = 0$, $r(x) = (1 - x^2)^{-1/2}$, and $\lambda = n^2$.

For future reference, Table 8.3 lists $p(x)$, $q(x)$, $r(x)$, and λ for the preceding equations, together with three other named equations that find applications in numerical analysis and elsewhere.

TABLE 8.3 $p(x), q(x), r(x)$ and λ for Some Named Equations

Name	$p(x)$	$q(x)$	$r(x)$	λ
Simple harmonic equation	1	0	1	n^2
Legendre's equation	$1-x^2$	0	1	$\alpha(\alpha+1)$
Bessel's equation	x	$-\nu^2/x$	x	k^2
Bessel's modified equation	x	$-\nu^2/x$	$-x$	k^2
Laguerre equation	xe^{-x}	0	e^{-x}	n
Chebyshev equation	$(1-x^2)^{1/2}$	0	$(1-x^2)^{-1/2}$	n^2
Hermite equation	e^{-x^2}	0	e^{-x^2}	$2n$

When the Sturm–Liouville equation (122) is associated with boundary conditions at $x = a$ and $x = b$, the equation itself together with the boundary conditions form what is called a **Sturm–Liouville problem**. The boundary conditions that will concern us here are the **homogeneous** boundary conditions,

$$A_1 y(a) + A_2 y'(a) = 0 \quad \text{and} \quad B_1 y(b) + B_2 y'(b) = 0, \tag{127}$$

where the term *homogeneous* is used in the sense that the linear combinations of $y(x)$ and $y'(x)$ at $x = a$ and $x = b$ are both equal to zero. There are three categories of Sturm–Liouville problems that arise, called **regular**, **periodic**, and **singular** problems according to the nature of the boundary conditions and the behavior of $p(x)$ at the boundaries.

Regular Sturm–Liouville problems

Regular problems are those for which constant values of λ are sought corresponding to each of which a nontrivial solution can be found for the Sturm–Liouville equation

$$(py')' + (q + \lambda r)y = 0,$$

with $p(x) > 0$ continuous on $a \le x \le b$ and subject to the boundary conditions

$$A_1 y(a) + A_2 y'(a) = 0 \quad \text{and} \quad B_1 y(b) + B_2 y'(b) = 0,$$

where in neither of the boundary conditions do both constant coefficients vanish.

Periodic Sturm–Liouville problems

This class of problems arises when $p(x)$ and the boundary conditions involving $y(x)$ and $y'(x)$ are periodic over the interval $a \le x \le b$. In this case constant values of λ are sought corresponding to each of which a nontrivial solution can be found for the Sturm–Liouville problem

$$(py')' + (q + \lambda r)y = 0,$$

subject to the periodic boundary conditions

$$p(a) = p(b), \quad y(a) = y(b), \quad \text{and} \quad y'(a) = y'(b).$$

Singular Sturm–Liouville problems

In this class of problems constant values of λ are sought, corresponding to each of which a nontrivial solution can be found for the Sturm–Liouville equation

$$(py')' + (q + \lambda r)y = 0,$$

on a finite interval at one or or both ends of which $p(x)$ or $r(x)$ vanish, or on a semi-infinite or infinite interval. The most frequently occurring problem of this type, and the only one to be considered here, is the Sturm–Liouville problem defined on a finite interval $a \le x \le b$, where the singular point is located at either $x = a$ or $x = b$, so that either $p(a) = 0$ or $p(b) = 0$. In such cases the boundary condition that is often imposed at the singular point takes the form of the requirement that the solution remains bounded there. Typically, this happens when a bounded solution of Bessel's equation of the form $y(x) = AJ_0(x) + BY_0(x)$ is required over an interval $0 \le x \le a$, because then the requirement that the solution remains bounded at the singular point located at $x = 0$ means we must set $B = 0$ to exclude the infinite value of $Y_0(x)$ at $x = 0$.

When dealing with Sturm–Liouville problems, each value of λ for which a nontrivial solution can be found is called an **eigenvalue** of the problem, and the corresponding solution $y(x)$ is called an **eigenfunction** of the problem. Because the Sturm–Liouville equation (122) is homogeneous, it follows that an eigenfunction can be multiplied by any constant factor and still remain an eigenfunction. This simple but fundamental property will be used repeatedly, first when normalizing eigenfunctions and later when representing arbitrary functions defined over an interval $[a, b]$ in terms of series of eigenfunctions, as is done in Chapter 9 when working with Fourier series. Such representations of functions are called *eigenfunction expansions*.

In most practical situations an eigenvalue is associated with an important physical characteristic of the problem, such as the frequency of vibration of a string or of a metal plate. In such cases the eigenfunction can be considered to describe a "snapshot" of a particular mode of vibration of the string or plate when it vibrates at the frequency determined by the associated eigenvalue. This application, and others that lead to Sturm–Liouville problems, will be developed in detail when partial differential equations are discussed in the context of *separation of variables*.

A Regular Problem

EXAMPLE 8.18 Find the eigenvalues and eigenfunctions of the two-point boundary value problem

$$y'' + \lambda y = 0,$$

such that

$$y(0) = 0 \quad \text{and} \quad y'(\pi) = 0.$$

Solution The interval over which the eigenfunctions are defined is $0 \le x \le \pi$. We need to consider the three cases $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$. The homogenous boundary conditions in this problem are of the type given in (127) with $A_2 = 0$ and $B_1 = 0$, where the values of the constants A_1 and B_2 are immaterial provided neither is zero.

***Case* $\boldsymbol{\lambda = 0}$**

When $\lambda = 0$ the equation has the general solution

$$y(x) = C_1 x + C_2,$$

so to satisfy the boundary condition $y(0) = 0$ we must have $C_2 = 0$, and to satisfy the boundary condition $y'(\pi) = 0$ we must have $C_1 = 0$, giving rise to the trivial solution $y(x) \equiv 0$. Thus, $\lambda = 0$ is not an eigenvalue of the problem.

***Case* $\boldsymbol{\lambda < 0}$**

If we set $\lambda = -\mu^2$, the general solution becomes

$$y(x) = C_1 e^{\mu x} + C_2 e^{-\mu x},$$

so the imposition of the boundary conditions requires that

$$0 = C_1 + C_2 \quad \text{and} \quad 0 = \mu C_1 e^{\mu\pi} - \mu C_2 e^{-\mu\pi}.$$

After the elimination of C_2, this last result can be written

$$0 = 2\mu C_1 \cosh \mu\pi,$$

but $\mu > 0$, so as $\cosh \mu\pi \neq 0$, this is only possible if $C_1 = 0$, so again we obtain the trivial solution showing that the problem has no negative eigenvalues.

***Case* $\boldsymbol{\lambda > 0}$**

As $\lambda > 0$, it is convenient to set $\lambda = \mu^2$, when the general solution of the equation becomes

$$y(x) = C_1 \cos \mu x + \mu C_2 \sin \mu x.$$

Applying the boundary condition $y(0) = 0$ to the general solution gives $C_1 = 0$, and applying the boundary condition $y'(\pi) = 0$ gives

$$\mu C_2 \cos \mu\pi = 0,$$

so either $C_2 = 0$ or $\cos \mu\pi = 0$. If we take $C_2 = 0$, then as $C_1 = 0$ we obtain the trivial solution, so we must take $C_2 \neq 0$. The condition $\cos \mu\pi = 0$ is satisfied if $\mu\pi$ is one of the zeros of the cosine function given by $\pm\frac{1}{2}(2n+1)\pi$, for $n = 0, 1, 2, \ldots$.

Denoting the permitted values of μ by μ_n we arrive at the condition

$$\mu_n = \pm\frac{1}{2}(2n+1), \quad \text{with } n = 0, 1, 2, \ldots.$$

The eigenvalues of this problem corresponding to the parameter $\lambda = \mu^2$ are thus

$$\lambda_n = \frac{(2n+1)^2}{4}, \quad \text{with } n = 0, 1, 2, \ldots,$$

and the corresponding eigenfunctions are

$$y_n(x) = \sin \frac{(2n+1)x}{2} \quad \text{with } n = 0, 1, 2, \ldots.$$

When writing down the form of the eigenfunction $y_n(x)$, we have set $C_2 = 1$ because, as has already been remarked, an eigenfunction can be multiplied by any constant nonzero factor and still remain an eigenfunction.

This example has shown the existence of an infinite increasing sequence of positive eigenvalues μ_n^2, corresponding to each of which there is a nontrivial solution of the Sturm–Liouville problem, namely the eigenfunction $y_n(x) = \sin \mu_n x$. If $\mu \neq \mu_n$, then the Sturm–Liouville problem only has the *trivial* solution $y(x) \equiv 0$. ■

A Periodic Problem

EXAMPLE 8.19 Find the eigenvalues and eigenfunctions of the Sturm–Liouville equation

$$y'' + \lambda y = 0$$

subject to the conditions

$$y(0) = y(L), \quad y'(0) = y'(L).$$

Solution The interval over which the eigenfunctions are defined is $0 \leq x \leq L$, and as in Example 8.18 we must again consider the three cases $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$.

Case $\boldsymbol{\lambda = 0}$

As in the previous problem, the general solution is

$$y(x) = C_1 x + C_2,$$

so applying the boundary condition $y(0) = y(L)$ leads to the result $C_2 = C_1 L + C_2$, from which it follows that $C_1 = 0$. As $y'(x) = C_1$ the boundary condition $y'(0) = y'(L)$ is automatically satisfied, showing that $y(x) = C_2$, with C_2 any nonzero constant. This shows that in this case $\lambda = 0$ is an eigenvalue, and that $y(x) = C_2$ (C_2 is an arbitrary nonzero constant) is the corresponding eigenfunction.

Case $\boldsymbol{\lambda < 0}$

If we set $\lambda = -\mu^2$, the general solution becomes

$$y(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}.$$

The boundary condition $y(0) = y(L)$ leads to the condition

$$C_1(1 - e^{\mu L}) = C_2(e^{-\mu L} - 1),$$

and the boundary condition $y'(0) = y'(L)$ leads to the condition

$$C_1(1 - e^{\mu L}) = -C_2(e^{-\mu L} - 1).$$

This last condition is only possible if $C_1 = 0$, but then $C_2 = 0$, so we again obtain the trivial solution. Consequently, we conclude that this problem has no negative eigenvalues.

Case $\boldsymbol{\lambda > 0}$

Setting $\lambda = \mu^2$ the general solution of the equation becomes

$$y(x) = C_1 \cos \mu x + C_2 \sin \mu x.$$

The boundary condition $y(0) = y(L)$ leads to the condition

$$C_1(1 - \cos \mu L) = C_2 \sin \mu L,$$

and the boundary condition $y'(0) = y'(L)$ leads to the condition

$$C_2(1 - \cos \mu L) = -C_1 \sin \mu L.$$

Eliminating C_2 between these two equations and simplifying the result gives

$$2C_1(1 - \cos \mu L) = 0.$$

This condition is satisfied if either $C_1 = 0$, or if $\cos \mu L = 1$. If $C_1 = 0$, then $C_2 = 0$, and we obtain the trivial solution, so the only other possibility is that $\cos \mu L = 1$. This last condition will be satisfied if μL is zero or an integer multiple of 2π, so

$$\mu L = \pm 2n\pi \quad \text{for } n = 0, 1, 2, \ldots,$$

or

$$\mu_n = \pm 2n\pi/L \quad \text{for } n = 0, 1, 2, \ldots.$$

As $\lambda = \mu^2$ the eigenvalues are seen to be

$$\lambda_n = 4n^2\pi^2/L^2, \quad \text{for } n = 0, 1, 2, \ldots.$$

The corresponding eigenfunctions are

$$y_n(x) = C_1 \cos \mu_n x + C_2 \sin \mu_n x,$$

or

$$y_n(x) = C_1 \cos(2n\pi x/L) + C_2 \sin(2n\pi x/L), \quad \text{for } n = 0, 1, 2, \ldots,$$

where not both constants C_1 and C_2 are zero. Because C_1 and C_2 are arbitrary, and both the cosine function and the sine function satisfy the Sturm–Liouville equation and the boundary conditions, by first setting $C_1 = 1$ and $C_2 = 0$ and then $C_1 = 0$ and $C_2 = 1$ it is seen that in this case the *single* eigenvalue $\lambda_n = 4n^2\pi^2/L^2$ has associated with it the *two* distinct eigenfunctions

$$y_n^{(1)}(x) = \cos(2n\pi x/L) \quad \text{and} \quad y_n^{(2)}(x) = \sin(2n\pi x/L).$$ ■

The eigenvalues in Sturm–Liouville problems are not always determined as easily as in the previous examples, and this is illustrated by the next example.

EXAMPLE 8.20 Find the eigenvalues and eigenfunctions of the Sturm–Liouville equation

$$y'' + \lambda y = 0,$$

subject to the conditions

$$y(0) - y'(0) = 0, \quad y(1) + y'(1) = 0.$$

Solution The interval over which the eigenfunctions are defined is $0 \le x \le 1$, and as before we must again consider the cases $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$.

Case $\lambda = 0$

The general solution is

$$y(x) = C_1 x + C_2,$$

so applying the boundary condition $y(0) - y'(0) = 0$ shows that $C_2 - C_1 = 0$, while applying the boundary condition $y(1) + y'(1) = 0$ gives the condition $2C_1 + C_2 = 0$.

The only solution for these equations is $C_1 = C_2 = 0$ corresponding to the trivial solution, so $\lambda = 0$ is not an eigenvalue of the problem.

***Case* $\boldsymbol{\lambda < 0}$**

Setting $\lambda = -\mu^2$ leads to the general solution

$$y(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}.$$

Applying the boundary condition $y(0) - y'(0) = 0$ leads to the condition

$$C_1(1-\mu) + C_2(1+\mu) = 0,$$

and applying the boundary condition $y(1) + y'(1) = 0$ leads to the condition

$$C_1[(1+\mu)e^{\mu} + C_2(1-\mu)e^{-\mu}] = 0.$$

As a factor $\mu - 1$ appears, we must consider the cases $\mu = 1$ and $\mu \neq 1$ separately. If $\mu = 1$, the first equation gives $C_2 = 0$, and the second one gives $C_1 = 0$, corresponding to the trivial solution. So $\mu = 1$ is not an eigenvalue. If $\mu \neq 1$, eliminating C_2 between these two equations leads to the condition

$$C_1[(1+\mu)^2 e^{\mu} - (1-\mu)^2 e^{-\mu}] = 0.$$

As $\mu > 0$, $(\mu+1)^2 e^{\mu} > (\mu-1)^2 e^{-\mu}$, showing that the bracketed term is nonvanishing, from which we conclude that $C_1 = 0$, and so $C_2 = 0$, corresponding to the trivial solution. Thus, this Sturm–Liouville problem has no negative eigenvalues.

***Case* $\boldsymbol{\lambda > 0}$**

Setting $\lambda = \mu^2$ leads to the general solution

$$y(x) = C_1 \cos \mu x + C_2 \sin \mu x.$$

Applying the boundary condition $y(0) - y'(L) = 0$ shows that

$$C_1 - \mu C_2 = 0,$$

and applying the boundary condition $y(1) + y'(1) = 0$ gives

$$C_1 \cos \mu + C_2 \sin \mu - \mu C_1 \sin \mu + \mu C_2 \cos \mu = 0.$$

Eliminating C_1 between these two equations, we obtain

$$C_2[2\mu \cos \mu + (1-\mu^2) \sin \mu] = 0.$$

The constant C_2 cannot be zero, because then $C_1 = 0$, corresponding to the trivial solution, so μ must be a solution of the equation

$$2\mu \cos \mu + (1-\mu^2) \sin \mu = 0$$

or, equivalently, μ_n is a solution of the transcendental equation

$$\tan \mu_n = \frac{2\mu_n}{\mu_n^2 - 1}.$$

This equation can only be solved numerically, but approximate solutions can be found graphically. Figure 8.12(a) shows graphs of $y = \tan \mu$ and $y = 2\mu/(\mu^2 - 1)$, and the required solutions μ_n are the values of μ at which the graphs intersect. It has been shown that $\mu = 1$ is not an eigenvalue, so the permissible values of μ_n are all greater than 1. The vertical lines to the right of $x = 1$ are the asymptotes to the

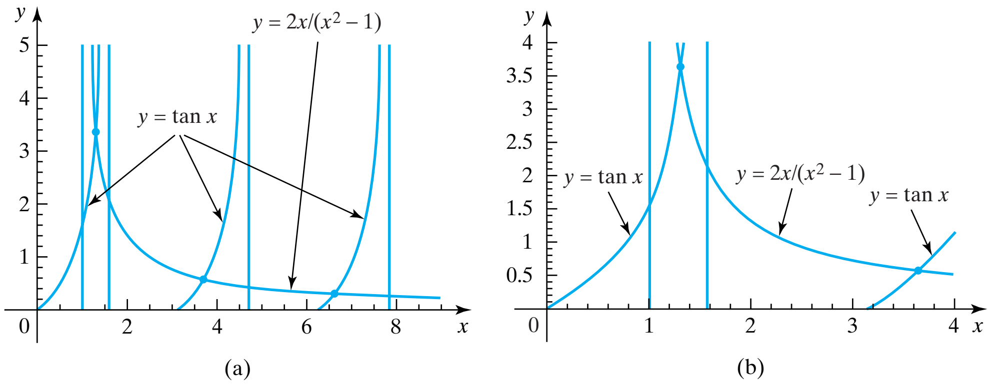


FIGURE 8.12 The roots of $\tan\mu = 2\mu/(\mu^2-1)$.

tangent function, and the vertical line at $x = 1$ is the asymptote to $2x/(x^2-1)$, to the right of which must lie all the solutions μ_n. The graph in Fig. 8.12b, drawn on a larger scale, shows that the first two values of μ are approximately $\mu_1 = 1.3$ and $\mu_2 = 3.7$. A numerical calculation using Newton's method described in Chapter 19 gives the better approximations $\mu_1 = 1.30654$ and $\mu = 3.67319$. It can be seen from Fig. 8.12a that when n is large $\mu_n \approx n\pi$. ■

A Singular Problem

EXAMPLE 8.21

Find the eigenvalues and eigenfunctions of Bessel's equation

$$x^2y'' + xy' + (k^2x^2 - n^2)y = 0$$

on the interval $0 \le x \le a$ on which the solution is bounded with $y(a) = 0$.

Solution This is a singular Sturm–Liouville problem, because when Bessel's equation is written in the Sturm–Liouville form

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right] + \left(k^2x^2 - \frac{n^2}{x}\right)y = 0,$$

with $p(x) = x$, $q(x) = -n^2/x$, $r(x) = x$, and $\lambda = k^2$ (see Table 8.3), it is seen that $p(0) = 0$.

The general solution is

$$y(x) = C_1J_n(kx) + C_2Y_n(kx),$$

but $Y_n(kx)$ is infinite when $x = 0$, so for the solution to remain finite over the interval $0 \le x \le a$ we must set $C_2 = 0$.

The solution now reduces to

$$y(x) = C_1J_n(kx),$$

so if the boundary condition $y(a) = 0$ is to be satisfied we must set

$$J_n(ka) = 0.$$

This condition will be satisfied if ka is one of the zeros of $J_n(x)$. If we denote the zeros of $J_n(x)$ by $j_{n,r}$, with $r = 1, 2, \ldots,$ it follows that k must be such that it assumes

one of the values

$$k_n = j_{n,r}/a, \quad \text{with } r = 1, 2, \ldots .$$

Thus, the eigenvalues $\lambda_n = k_n^2$ are given by

$$\lambda_n = j_{n,r}^2/a^2,$$

and the corresponding eigenfunctions are

$$y_r(x) = J_n(j_{n,r}x/a), \quad \text{with } r = 1, 2, \ldots,$$

where for convenience we have set $C_1 = 1$. Table 8.1 lists the first six zeros of $J_n(x)$ for $n = 0, 1, 2, 3$. Thus if, for example, we consider the case $n = 0$, the corresponding zeros are seen to be $j_{0,1} = 2.4048$, $j_{0,2} = 5.5201\ldots$, so the eigenvalues are $\lambda_1 = 5.7832/a^2$, $\lambda_2 = 30.4711/a^2, \ldots$, and the corresponding eigenfunctions are

$$y_1(x) = J_0(2.4048x/a), \quad y_2(x) = J_0(5.5201x/a), \ldots .$$

■

Orthogonal and Orthonormal Systems of Functions

orthogonal and orthonormal systems

When working with eigenfunctions it is useful to introduce the notions of **orthogonal** and **orthonormal** systems of eigenfunctions that are defined as follows.

Let $\varphi_1(x), \varphi_2(x), \ldots$ be an infinite sequence of functions defined over the interval $a \leq x \leq b$ on which a function $r(x) \geq 0$ is defined. Then the functions are said to be **orthogonal** with respect to the **weight function** $r(x)$ if

$$\int_a^b r(x)\varphi_m(x)\varphi_n(x)dx = 0 \quad \text{for } m \neq n.$$

Clearly, the integral $\int_a^b r(x)\varphi_m(x)\varphi_n(x)dx > 0$ when $m = n$, so we can define a number $\|\varphi_n(x)\|$, called the **norm** of $\varphi_n(x)$, where the square of the norm is defined as

$$\|\varphi_n(x)\|^2 = \int_a^b r(x)\varphi_n^2(x)dx.$$

Using this definition of the norm it is easy to see that the sequence of normalized functions $\hat{\varphi}_1(x) = \varphi_1(x)/\|\varphi_1(x)\|$, $\hat{\varphi}_2(x) = \varphi_2(x)/\|\varphi_2(x)\|, \ldots$ has the property that

$$\int_a^b \hat{\varphi}_m(x)\hat{\varphi}_n(x)r(x)dx = 0, \quad \text{for } m \neq n$$

and

$$\int_a^b \hat{\varphi}_m(x)\hat{\varphi}_n(x)r(x)dx = 1, \quad \text{for } m = n.$$

The sequence of functions $\hat{\varphi}_1(x), \hat{\varphi}_2(x), \ldots$ derived from the sequence of orthogonal functions $\varphi_1(x), \varphi_2(x), \ldots$ by normalization is said to form an **orthonormal** sequence of functions.

In what follows the orthogonality of eigenfunctions will be used extensively, but for the moment it will be sufficient to give a single elementary example of an orthogonal sequence of functions.

EXAMPLE 8.22 Show that the sequence of functions

$$1,\ \cos x,\ \sin x,\ \cos 2x,\ \sin 2x,\ \cos 3x,\ \sin 3x, \ldots$$

is orthogonal over the interval $-\pi \leq x \leq \pi$ with respect to the weight function $r(x) = 1$, and use it to construct an orthonormal sequence.

Solution The functions in this sequence occur in the Fourier series representation of an arbitrary function $f(x)$ defined over the interval $-\pi \leq x \leq \pi$ that is discussed in Chapter 9. Routine calculation shows that for $m \neq n$,

$$\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = 0, \quad \int_{-\pi}^{\pi} \cos mx \cos nx\, dx = 0, \quad \int_{-\pi}^{\pi} \sin mx \cos nx\, dx = 0,$$

and

$$\int_{-\pi}^{\pi} 1 dx = 2\pi, \quad \int_{-\pi}^{\pi} \sin^2 nx dx = \int_{-\pi}^{\pi} \cos^2 nx dx = \pi, \quad n = 1, 2, \ldots,$$

while $\int_{-\pi}^{\pi} 1 \cdot \cos mx dx = \int_{-\pi}^{\pi} 1 \cdot \sin mx dx = 0.$

So the functions

$$1,\ \cos x,\ \sin x,\ \cos 2x,\ \sin 2x,\ \cos 3x,\ \sin 3x, \ldots$$

are orthogonal over the interval $-\pi \leq x \leq \pi$ with respect to the weight function $r(x) = 1$. The respective norms are $\|1\| = \sqrt{2\pi}$ and $\|\sin nx\| = \|\cos nx\| = \sqrt{\pi}$, so the sequence of functions

$$1/\sqrt{2\pi}, \quad (\sin nx)/\sqrt{\pi}, \quad (\cos nx)/\sqrt{\pi}, \quad \text{with } n = 1, 2, \ldots,$$

forms an orthonormal sequence. ■

Fundamental Properties of Eigenvalues

The theorem that follows lists the most important properties of the eigenvalues and eigenfunctions of Sturm–Liouville problems. Apart from the important Rayleigh quotient that occurs in Theorem 8.3 (5), the other properties are all qualitative and their main use is to provide general information about eigenvalues that is often of considerable value when working with physical problems.

For convenience, the proofs of all results in Theorem 8.3 that can be established in a straightforward manner have been included in an appendix at the end of this chapter. The proofs of the other results can be found in the references listed at the end of the chapter. A reader who does not require the proofs that are given here may omit them, though the properties themselves should be understood.

THEOREM 8.3 **A Sturm–Liouville theorem**

1. Regular and periodic Sturm–Liouville problems have an infinite number of distinct real eigenvalues $\lambda_1, \lambda_2, \ldots,$ that can be arranged in order so that

important properties of eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots,$$

where the smallest eigenvalue λ_1 is finite, and

$$\lim_{n\to\infty} \lambda_n = \infty.$$

2. To each eigenvalue of a regular Sturm–Liouville problem there corresponds only one eigenfunction that is unique apart from an arbitrary multiplicative constant.
3. Let the eigenfunctions of a Sturm–Liouville problem on an interval $a \le x \le b$ with weight function $r(x)$ be denoted by $\varphi_1, \varphi_2, \ldots,$ with the corresponding eigenvalues $\lambda_1, \lambda_2, \ldots.$ Then, if φ_m and φ_n are eigenfunctions corresponding to two distinct eigenvalues λ_m and λ_n ($\lambda_m \ne \lambda_n$ for $m \ne n$), the functions are orthogonal with respect to the weight function $r(x)$, so

$$\int_a^b r(x)\varphi_m(x)\varphi_n(x)dx = 0.$$

4. All the eigenvalues of a Sturm–Liouville problem are real.
5. Let λ_n be an eigenvalue of a regular Sturm–Liouville problem, with φ_n its associated eigenfunction defined on an interval $a \le x \le b$. Then λ_n is given in terms of the Sturm–Liouville functions $p, q, r,$ and the boundary conditions by the **Rayleigh quotient**

$$\lambda_n = \frac{-[p\varphi_n\varphi_n']_a^b + \int_a^b p(\varphi_n')^2 dx - \int_a^b q\varphi_n^2\, dx}{\int_a^b r\varphi_n^2\, dx}.$$

6. Let λ_n be an eigenvalue and φ_n be the corresponding eigenfunction of a regular Sturm–Liouville problem defined on $a \le x \le b$. Then if $q(x) < 0$ and $[p(x)\varphi_n\varphi_n']_a^b \le 0$, all the eigenvalues are nonnegative.
7. The nth eigenfunction of a regular Sturm–Liouville problem defined on the interval $a \le x \le b$ has exactly $n - 1$ zeros lying strictly inside the interval.
8. Let two regular Sturm–Liouville problems defined on an interval $a \le x \le b$ be such that $[p(x)\varphi_n\varphi_n']_a^b = 0$ and differ only in their coefficients $p(x)$. Furthermore, let the problem with the coefficient $p_1(x)$ have the eigenvalues $\lambda_1^{(1)}, \lambda_2^{(1)}, \ldots,$ and the problem with the coefficient $p_2(x)$ have the eigenvalues $\lambda_1^{(2)}, \lambda_2^{(2)}, \ldots.$ Then, if $p_1(x) > p_2(x)$,

$$\lambda_n^{(1)} > \lambda_n^{(2)} \quad \text{for } n = 1, 2, \ldots.$$

9. Let a regular Sturm–Liouville equation with $q(x) < 0$ be defined on an interval $a \le x \le b$ and have boundary conditions such that the first term in the numerator of the Rayleigh quotient in Property 5 is zero. Then reducing the length of the interval $a \le x \le b$ will not reduce the value of any eigenvalue. ■

Remarks about Theorem 8.3

Property 1 ensures that the eigenvalues are distinct ($\lambda_m \ne \lambda_n$ if $m \ne n$), that they are infinite in number, and, because $\lim_{n\to\infty} \lambda_n = \infty$, that there can be no clustering

of eigenvalues about a finite limit point. If, for example, the eigenvalues represent the frequencies of vibration of a stretched string of finite length L, this means there is a lowest frequency of vibration, but no upper limit to the frequency of vibration of the string.

Property 2 says that to each distinct eigenvalue of a regular Sturm–Liouville problem there corresponds only one eigenfunction, and it is unique apart from a constant multiplicative factor. Notice that this only applies to *regular* Sturm–Liouville problems, because in periodic Sturm–Liouville problems an eigenvalue has associated with it two linearly independent eigenfunctions. This latter situation occurred in Example 8.19, where the *two* eigenfunctions

$$y_n^{(1)}(x) = \cos(2n\pi x/L) \quad \text{and} \quad y_n^{(2)}(x) = \sin(2n\pi x/L)$$

were seen to correspond to the *single* eigenvalue $\lambda_n = 4n^2\pi^2/L^2$. In such cases there can only be two eigenfunctions to each eigenvalue, because the equation is second order. The scaling of eigenfunctions by a constant is used repeatedly when representing arbitrary functions in terms of series of eigenfunctions.

Property 3 is of fundamental importance because of the part played by orthogonality when developing arbitrary functions in terms of series of eigenfunctions defined over some interval. It is the orthogonality of sine and cosine functions illustrated in Example 8.22 that is used when working with Fourier series.

It will be seen later that the representation (*expansion*) of arbitrary functions in terms of series of eigenfunctions is more general than in terms of power series. This is because, unlike Taylor series whose coefficients are determined by repeated differentiation of the function being expanded, the coefficients in series of eigenfunctions are determined in terms of integrals involving the function. This means that the function can have finite discontinuities at points within its interval of representation and still have an eigenfunction expansion.

Property 4 removes the necessity to check Sturm–Liouville problems for the possibility that negative eigenvalues occur. Had this property been known in advance of Examples 8.18 to 8.21, it would have been unnecessary to have examined the forms of solution corresponding to $\lambda < 0$.

Property 5 is useful when seeking qualitative properties of eigenvalues. The result is not directly useful when trying to determine an eigenvalue because the associated eigenfunction needs to be known. The main use of the Rayleigh quotient arises when it is used in the following rather different form.

Let a function $\Phi(x)$ containing some arbitrary constants $\alpha, \beta, \ldots$ satisfy the *boundary conditions* of a Sturm–Liouville problem. Then with any choice of the arbitrary constants, the Rayleigh quotient

$$\frac{-[p\Phi_n\Phi_n']_a^b + \int_a^b p(\Phi_n')^2\,dx - \int_a^b q\Phi_n^2\,dx}{\int_a^b r\Phi_n^2\,dx} \tag{128}$$

provides an *upper bound* for the value of the smallest eigenvalue of the associated Sturm–Liouville problem. If the arbitrary constants $\alpha, \beta, \ldots$ are chosen to *minimize* this expression, its value becomes the best estimate of the smallest eigenvalue that can be obtained using that approximation. Furthermore, substituting the values of the constants that minimize the Rayleigh quotient into the function $\Phi(x)$ provides a corresponding approximation to the first eigenfunction. The actual value λ_1 is only attained when $\Phi(x) = \varphi_1(x)$.

Property 6, together with Property 4, ensures that under the given conditions the eigenvalues are both real and positive. In corresponding physical problems

this result is usually to be expected on an intuitive basis, so the result provides the mathematical justification for making such an assumption on purely physical grounds.

Property 7 provides precise information about the number of zeros of a given eigenfunction within the interval over which it is defined. It is well illustrated by considering Figs. 8.1 showing graphs of Legendre polynomials. These show, for example, that $P_3(x)$ has precisely three zeros in the interval $-1 \le x \le 1$, whereas $P_4(x)$ has four zeros. It is important to recognize that these zeros lie strictly *inside* the interval, so that zeros that occur at either end of an interval are *not* counted.

Property 8 means that if in a Sturm–Liouville problem $p(x)$ is associated with a characteristic feature of a physical system, then increasing $p(x)$ increases each eigenvalue of the system. For example, if $p(x)$ is related to the density of a vibrating string, then *increasing* the density while keeping all other parameters constant will *decrease* the frequency of vibration, and increasing the tension will increase the frequency.

Property 9 means that reducing the length of the interval $a \le x \le b$ on which a Sturm–Liouville problem is set cannot reduce the values of the eigenvalues. In fact, it usually increases them. This is most easily understood in terms of a vibrating string for which the eigenvalues of the associated Sturm–Liouville problem represent its possible frequencies of vibration (see Chapter 18). In such a case *shortening* the string, while leaving other parameters unchanged, will *increase* the frequency, as any guitarist or violinist knows from experience.

EXAMPLE 8.23

orthogonality and weight functions

An orthogonal system of sine functions The Sturm–Liouville problem considered in Example 8.18, namely

$$y'' + \lambda y = 0 \quad \text{with } y(0) = 0 \quad \text{and} \quad y'(\pi) = 0,$$

is such that $p(x) = 1$, $q(x) = 0$, and $r(x) = 1$. Its eigenvalues were shown to be $\lambda_n = (2n+1)^2/4$, and its corresponding eigenfunctions were

$$\varphi_n(x) = \sin\frac{(2n+1)x}{2}, \quad n = 0, 1, \ldots.$$

Thus, from Theorem 8.3 (3), the functions $\varphi_n(x)$ are orthogonal over the interval $0 \le x \le \pi$ with weight function $r(x) = 1$, and so

$$\int_0^\pi \varphi_m(x)\varphi_n(x)dx = 0 \quad \text{for } m \ne n.$$

The square of the norm is given by

$$\|\varphi_n(x)\|^2 = \left\|\sin\frac{(2n+1)x}{2}\right\|^2 = \int_0^\pi \left(\sin\frac{(2n+1)x}{2}\right)^2 dx = \frac{\pi}{2},$$

so $\|\varphi_n(x)\| = \sqrt{\pi/2}$. ■

EXAMPLE 8.24

Orthogonality of Legendre polynomials When written in Sturm–Liouville form, Legendre's equation becomes

$$[(1-x^2)y']' + \lambda y = 0,$$

and it is defined over the interval $-1 \le x \le 1$, with $p(x) = 1 - x^2$, $q(x) = 0$, and $r(x) = 1$. The Legendre polynomial $P_n(x)$ corresponds to $\lambda = n(n+1)$, so from

Theorem 8.3 (3) we see that the Legendre polynomials are orthogonal with respect to the weight function $r(x) = 1$, so that

$$\int_{-1}^{1} P_m(x)P_n(x)dx = 0 \quad \text{for } m \neq n.$$

To determine the norm $\|P_n(x)\|$ we make use of recurrence relation (16),

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0.$$

Replacing n by $n-1$ and substituting for one of the factors $P_n(x)$ in the integral gives

$$\begin{aligned}
\|P_n(x)\|^2 &= \int_{-1}^{1} P_n(x)\left\{\left(\frac{2n-1}{n}\right)xP_{n-1}(x) - \frac{n-1}{n}P_{n-2}(x)\right\}dx \\
&= \left(\frac{2n-1}{n}\right)\int_{-1}^{1} xP_{n-1}(x)P_n(x)dx - \left(\frac{n-1}{n}\right)\int_{-1}^{1} P_n(x)P_{n-2}(x)dx \\
&= \left(\frac{2n-1}{n}\right)\int_{-1}^{1} xP_{n-1}(x)P_n(x)dx,
\end{aligned}$$

where the second integral has been set equal to zero because of the orthogonality of $P_n(x)$ and $P_{n-2}(x)$. Using the recurrence relation to remove the term $xP_n(x)$ gives

$$\begin{aligned}
\|P_n(x)\|^2 &= \left(\frac{2n-1}{n}\right)\int_{-1}^{1} P_{n-1}(x)\left\{\left(\frac{n+1}{2n+1}\right)P_{n+1}(x) + \left(\frac{n}{2n+1}\right)P_{n-1}(x)\right\}dx \\
&= \left(\frac{2n-1}{2n+1}\right)\int_{-1}^{1} [P_{n-1}(x)]^2\,dx,
\end{aligned}$$

where the first integral vanishes because of the orthogonality of $P_n(x)$ and $P_{n-1}(x)$.

This has established the recurrence relation for norms

$$\|P_n(x)\|^2 = \left(\frac{2n-1}{2n+1}\right)\|P_{n-1}(x)\|^2.$$

Using this result to relate $\|P_n(x)\|^2$ to $\|P_0(x)\|^2$ and cancelling factors shows that

$$\begin{aligned}
\|P_n(x)\|^2 &= \left(\frac{2n-1}{2n+1}\right)\left(\frac{2n-3}{2n-1}\right)\left(\frac{2n-5}{2n-3}\right)\cdots\left(\frac{3}{5}\right)\left(\frac{1}{3}\right)\|P_0(x)\|^2 \\
&= \left(\frac{1}{2n+1}\right)\|P_0(x)\|^2,
\end{aligned}$$

but $\|P_0(x)\|^2 = \int_{-1}^{1} 1dx = 2$, so that

$$\|P_n(x)\|^2 = \frac{2}{2n+1}, \quad \text{and} \quad \|P_n(x)\| = \sqrt{\frac{2}{2n+1}} \quad \text{for } n = 0, 1, \ldots. \quad ■$$

EXAMPLE 8.25

Orthogonality of Bessel functions $J_n(x)$ When written in Sturm–Liouville form, Bessel's equation of order n becomes

$$[xJ_n'(kx)]' + \left(k^2x - \frac{n^2}{x}\right)J_n(kx) = 0,$$

where $p(x) = x$, $q(x) = -n^2/x$, $r(x) = x$, and $\lambda = k^2$.

The orthogonality of Bessel functions over an interval $0 \le x \le a$ takes a somewhat different form from that in the previous examples, because the orthogonality is between Bessel functions of the *same* order, but with *different* arguments, rather than between Bessel functions of different orders. If for fixed n the solution $J_n(kx)$ is required to satisfy the boundary condition

$$J_n(ka) = 0,$$

it follows, as in Example 8.21, that the permissible values of k are

$$k_r = j_{n,r}/a, \quad \text{with } r = 1, 2, \ldots,$$

where $j_{n,r}$ is the rth zero of $J_n(x)$, the first few of which are listed in Table 8.1.

Theorem 8.3 (3) then asserts that as the weight function $r(x) = x$, the orthogonality condition is

$$\int_0^a x J_n\left(\frac{j_{n,r}x}{a}\right) J_n\left(\frac{j_{n,s}x}{a}\right) dx = 0 \quad \text{for } r \neq s.$$

The square of the norm of $J_n(\frac{j_{n,r}x}{a})$ is

$$\left\| J_n\left(\frac{j_{n,r}x}{a}\right) \right\|^2 = \int_0^a x\left[J_n\left(\frac{j_{n,r}x}{a}\right)\right]^2 dx = \frac{a^2}{2}[J_{n+1}(j_{n,r})]^2.$$

A proof of this last result is given in Appendix 2 at the end of the chapter. ■

EXAMPLE 8.26 **Orthogonality of Chebyshev polynomials** When written in Sturm–Liouville form, the Chebyshev equation for the polynomial $T_n(x)$ of degree n becomes

$$[(1 - x^2)^{1/2} y']' + n^2(1 - x^2)^{-1/2} y = 0.$$

As the weight function is $(1 - x^2)^{-1/2}$, the orthogonality relation becomes

$$\int_{-1}^1 \frac{T_m(x)T_n(x)}{\sqrt{1 - x^2}} dx = 0 \quad \text{for } m \neq n.$$

The square of the norm of $T_n(x)$ is given by

$$\|T_n(x)\|^2 = \int_{-1}^1 \frac{[T_n(x)]^2}{\sqrt{1 - x^2}} dx$$

where $\|T_0(x)\|^2 = \pi$ and $\|T_n(x)\|^2 = \pi/2$ for $n = 1, 2, \ldots$. As it is inappropriate to include the proof of this result here, an outline proof is given in Exercise 31 at the end of the section.

Accounts of Sturm–Liouville systems are to be found in references [3.3] and [3.4] and in Chapter 5 of reference [3.7]. ■

Summary

The important idea of Sturm–Liouville systems was introduced, their relationship to eigenvalues and eigenfunctions was explained, and it was shown that the solutions of such systems comprise a system of functions that are orthogonal with respect to a suitable weight function. The examples of Sturm–Liouville systems that were given included trigonometric, Legendre, Chebyshev, and Bessel functions. Infinite sets of functions like these represent generalizations to an infinite dimensional space of the elementary notion of the orthogonality of vectors in the three-dimensional Euclidean space. The significance of the orthogonality of eigenfunctions will become clear later when arbitrary functions are expanded in terms of eigenfunctions.

EXERCISES 8.10

In Exercises 1 through 4, reduce the differential equation to Sturm–Liouville form by the method used when reducing equation (121) to the form in (122).

1. $xy'' + (1 - x)y' + \lambda y = 0.$

2. $y'' - 2xy' + \lambda y = 0.$

3. $(1 - x^2)y'' - xy' + \lambda y = 0.$

4. $(1 - x^2)^2 y'' - 2x(1 - x^2)y' + [\lambda(1 - x^2) - m^2]y = 0.$

In Equations 5 through 14 find the eigenvalues and eigenfunctions of the differential equation.

5. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0.$

6. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0.$

7. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(1) = 0.$

8. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(2\pi) = 0.$

9. $y'' + \lambda y = 0, y(0) = 0, y'(1) - 2y(1) = 0.$ Find numerical estimates for the first two eigenvalues.

10. $y'' + \lambda y = 0, y(0) = 0, y'(1) + y(1) = 0.$ Find numerical estimates for the first two eigenvalues.

11. $y'' + \lambda y = 0, \quad y(-1) = y(1), \quad y'(-1) = y'(1).$

12. $y'' + \lambda y = 0, \quad y(0) = y(1), \quad y'(0) = y'(1).$

13. $x^2 y'' + xy' + k^2 y = 0, \quad y(1) = 0, \quad y(4) = 0.$ (Hint: This is a Cauchy–Euler equation)

14. $x^2 y'' + xy' + 9k^2 y = 0, \quad y(1) = 0, \quad y'(2) = 0.$ (Hint: This is a Cauchy–Euler equation)

In Exercises 15 through 18, verify that the sets of functions are orthogonal over their stated intervals with the weight function $r(x) = 1$, and find their norms.

15. $\varphi_n(x) = \sin\left(\dfrac{n\pi x}{L}\right), \quad n = 1, 2, \ldots \ (0 \le x \le L).$

16. $\varphi_n(x) = \cos\left(\dfrac{(2n-1)\pi x}{2}\right), \quad n = 1, 2, \ldots \ (0 \le x \le 1).$

17. $\varphi_n(x) = \cos\left(\dfrac{n\pi x}{L}\right), \quad n = 1, 2, \ldots \ (0 \le x \le L).$

18. $\varphi_n(x) = \sin\left(\dfrac{(2n-1)\pi x}{4}\right), \ n = 1, 2, \ldots (0 \le x \le 2\pi).$

19.* It is known from Example 8.18 that the Sturm–Liouville problem

$$y'' + \lambda y = 0 \quad \text{with } y(0) = 0,\ y'(\pi) = 0$$

has for its first eigenvalue $\lambda_1 = 1/4$, and that the corresponding eigenfunction is $\varphi_1(x) = \sin x/2$. Verify that the function $\Phi(x) = x(2\pi - x)$ satisfies the boundary conditions for y. By using this expression in the form of the Rayleigh quotient given in (128), find the corresponding upper bound for λ_1 and compare it with the exact value. Why is it that replacing $\Phi(x)$ by $\Phi(x) = Cx(2\pi - x)$, where C is any nonzero constant, leaves the estimate of the upper bound unchanged?

20.* Perform the calculation required in Exercise 19 using the function $\Phi(x) = x^2(1 - \frac{2x}{3\pi})$, after first showing that $\Phi(x)$ satisfies the boundary conditions. Compare the value of the upper bound so obtained with the exact value $\lambda_1 = 1/4$. Suggest a reason why this approximation is not likely to yield a better lower bound than the one obtained using the function $\Phi(x)$ in Exercise 19.

21.* The Sturm–Liouville form of Bessel's equation of order 1 is

$$[xy']' + \left(k^2 x - \frac{1}{x}\right) y = 0,$$

where $p(x) = x, q(x) = -1/x, r(x) = x$, and $\lambda = k^2$. The bounded solution of this equation on the interval $0 \le x \le 1$ subject to the condition $y(1) = 0$ is $y(x) = J_1(j_{1,1}x)$, where from Table 8.1 $j_{1,1} = 3.8317$ is the first zero of $J_1(x)$. The inverted parabola $\Phi(x) = x(1 - x)$ provides a reasonable approximation to the shape of the required Bessel function for $0 \le x \le 1$. Use this expression in (128) to obtain an upper bound for the first eigenvalue λ_1 of the equation, and using the fact that $\lambda_1 = j_{1,1}^2$ find an upper bound for $j_{1,1}$. Compare this estimate with the correct result.

22.* The Sturm–Liouville form of Bessel's equation of order 2 is

$$[xy']' + \left(k^2 x - \frac{4}{x}\right) y = 0,$$

where $p(x) = x$, $q(x) = -4/x$, $r(x) = x$, and $\lambda = k^2$. The solution of this equation that is bounded on the interval $0 \le x \le 1$ and subject to the condition $y(1) = 0$ is $y(x) = J_2(j_{2,1}x)$, where from Table 8.1 $j_{2,1} = 5.1316$ is the first zero of $J_2(x)$. Use the approximation $\Phi(x) = x(1 - x)$ to obtain an upper bound for the first eigenvalue of the equation, and using the fact that $\lambda_1 = j_{2,1}^2$, find an upper bound for $j_{2,1}$. Compare this estimate with the correct value.

23. The differential equation

$$L[y] = P(x)y'' + Q(x)y' + R(x)y = 0$$

has associated with it the **adjoint differential equation** defined by

$$M[w] = [P(x)w]'' - [Q(x)w]' + R(x)w = 0.$$

A differential equation is said to be **self-adjoint** if the differential equation and its adjoint are of the same form. When this occurs, the differential operator common to both equations is also said to be self-adjoint.

(a) Show that Bessel's equation of order ν

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

is not self-adjoint.

(b) Find the value of α that makes the following equation self-adjoint

$$(\alpha \sin x)y'' + (\cos x)y' + 2y = 0.$$

24. Show that Legendre's equation

$$(1 - x^2)y'' - 2xy' - \lambda y = 0$$

is self-adjoint.

25. Show that Bessel's equation of order n in the form

$$x^2y'' + xy' - (x^2 - n^2)y = 0$$

is not self-adjoint, but that it becomes so when multiplied by $1/x$.

26. Show that the Hermite equation in the form

$$y'' - 2xy' + \lambda y = 0$$

is not self-adjoint, but that it becomes so when multiplied by $\exp[-x^2]$.

27. Show that the Chebyshev equation in the form

$$(1 - x^2)y'' - xy' + \lambda y = 0$$

is not self-adjoint, but that it becomes self-adjoint when multiplied by $(1 - x^2)^{-1/2}$.

28.* Let $u(x)$ and $v(x)$ be any two solutions of

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = 0$$

defined over the interval $a \le x \le b$. Prove **Abel's identity**

$$p(x)[u(x)v'(x) - u'(x)v(x)] = \text{constant}$$

for all x in the interval. As $p(x) \ne 0$ in regular Sturm–Liouville problems, what conclusion can be drawn from Abel's identity if (a) the constant is not zero and (b) the constant is zero?
(Hint: Multiply the equations for u and v by suitable factors, subtract them, and integrate the resulting equation over the interval $a \le t \le x$.)

29.* The Chebyshev polynomial $T_n(x)$ can be defined as

$$T_n(x) = \cos(n \arccos x), \quad n = 0, 1, \ldots.$$

Verify this by showing that this definition of $T_n(x)$ satisfies the Chebyshev differential equation

$$(1 - x^2)y'' - xy' + n^2y = 0.$$

30.* Let $y = T_n(x) = \cos(n \arccos x)$ and set $x = \cos\theta$. Use the fact that $y(\theta)$ satisfies the differential equation

$$\frac{d^2y}{d\theta^2} + n^2y = 0$$

together with a change of variable back from θ to x to show that this definition of $T_n(x)$ satisfies the Chebyshev equation

$$(1 - x^2)y'' - xy' + n^2y = 0.$$

31.* Show that if $y_n(\theta) = \cos n\theta$ then

$$\int_0^\pi [y_n(\theta)]^2 d\theta = \begin{cases} \pi, & n = 0 \\ \frac{1}{2}\pi, & n \ge 1 \end{cases}.$$

By changing back from the variable θ to x, where $x = \cos\theta$ and using the definition of $T_n(x)$ in Problem 30, show that the square of the norm of $T_n(x)$ is given by

$$\|T_n(x)\|^2 = \int_{-1}^1 \frac{[T_n(x)]^2}{\sqrt{1 - x^2}}dx = \begin{cases} \pi, & n = 0 \\ \frac{1}{2}\pi, & n \ge 1 \end{cases}.$$

8.11 Eigenfunction Expansions and Completeness

The orthogonality of a set of functions $\varphi_0(x), \varphi_1(x), \ldots$ over the interval $a \le x \le b$ with respect to a weight function $r(x)$ allows them to be used to expand (represent) a function $f(x)$ over that same interval in terms of the functions $\varphi_i(x)$ by expressing it as the series

$$f(x) = \sum_{m=0}^{\infty} a_m\varphi_m(x) = a_0\varphi_0(x) + a_1\varphi_1(x) + \ldots, \tag{129}$$

where $a_0, a_1, \ldots$ are constants called the **coefficients** of the expansion.

The representation of functions in this manner is used in approximation theory, in numerical analysis, and in the solution of partial differential equations by the method of *separation of variables* to be described later (see Chapter 18). A series

eigenfunction expansions

such as (129) is called a **generalized Fourier series** representation of $f(x)$ or, when the functions $\varphi_n(x)$ are eigenfunctions, an **eigenfunction expansion** of $f(x)$.

To see how the coefficients a_m in (129) are derived for a specific function $f(x)$, it is necessary to recall that

$$\int_a^b r(x)\varphi_m(x)\varphi_n(x)dx = 0, \quad m \neq n, \tag{130}$$

and

$$\|\varphi_n(x)\|^2 = \int_a^b r(x)[\varphi_n(x)]^2\,dx. \tag{131}$$

If the expansion (129) is multiplied by $r(x)\varphi_n(x)$ and the result is integrated over the interval $a \le x \le b$, the orthogonality condition (130) causes every term on the right for which $m \neq n$ to vanish, leaving only the term involving a_n, so using (131) enables the result to be written

$$\int_a^b r(x)\varphi_n(x)f(x)dx = a_n\int_a^b r(x)[\varphi_n(x)]^2dx = a_n\|\varphi_n(x)\|^2.$$

This has established that the coefficients a_n are given by the formula

$$a_n = \frac{\int_a^b r(x)\varphi_n(x)f(x)dx}{\|\varphi_n(x)\|^2}, \quad n = 0, 1, \ldots. \tag{132}$$

The term-by-term integration of series (129) leading to (132) requires justification, and this follows when the series is uniformly convergent.

Summary of Main Sets of Orthogonal Functions

1. Fourier series (see Chapter 9)

Interval of definition	$-\pi \le x \le \pi$
Set of functions	$\{1, \cos nx, \sin nx\}, n = 1, 2, \ldots$
Weight	$r(x) = 1$
Orthogonality	$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = 0, m \neq n$
	$\int_{-\pi}^{\pi} \sin mx \cos nx\,dx = 0,$
	$\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = 0, m \neq n$
	$\int_{-\pi}^{\pi} 1 \cdot \sin mx\,dx = 0$
	$\int_{-\pi}^{\pi} 1 \cdot \cos mx\,dx = 0$
Norms	$\|1\|^2 = 2\pi, \|\cos nx\|^2 = \pi, \|\sin nx\|^2 = \pi$

2. Legendre polynomials

Interval of definition	$-1 \le x \le 1$
Set of functions	$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{1}{2}(3x^2 - 1), \ldots$
Recurrence relation	$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$
Weight	$r(x) = 1$
Orthogonality	$\int_{-1}^{1} P_m(x)P_n(x)dx = 0, m \ne n$
Norm	$\lVert P_n(x)\rVert^2 = \dfrac{2}{2n+1}, n = 0, 1, \ldots.$

3. Bessel functions

Interval of definition	$0 \le x \le a$
Set of functions	There is a set of orthogonal functions for each fixed n: $J_n(j_{n,r}x/a), r = 1, 2, \ldots,$ with $j_{n,r}$ the nth zero of $J_n(x)$ (see Table 8.1)
Weight	$r(x) = x$
Orthogonality	$\int_0^a xJ_n(j_{n,r}x/a)J_n(j_{n,s}x/a)dx = 0, r \ne s$
Norm	$\lVert J_n(j_{n,r}x/a)\rVert^2 = \frac{1}{2}a^2[J_{n+1}(j_{n,r})]^2, r = 1, 2, \ldots$

4. Chebyshev polynomials

Interval of definition	$-1 \le x \le 1$
Set of functions	$T_0(x) = 1,\ T_1(x) = x,\ T_2(x) = 2x^2 - 1, \ldots$
Recurrence relation	$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0$
Weight	$(1 - x^2)^{-1/2}$
Orthogonality	$\int_{-1}^{1} \dfrac{T_m(x)T_n(x)}{\sqrt{1-x^2}}dx = 0, m \ne n$
Norms	$\lVert T_0(x)\rVert^2 = \pi,\ \lVert T_n(x)\rVert^2 = \frac{1}{2}\pi, \quad n = 1, 2, \ldots.$

(See Exercises 30 and 31 in Exercise Set 18.10 for the derivation of the norms.)

EXAMPLE 8.27

a first example of a Fourier series

A Fourier series Example 8.22 established the orthogonality of the set of functions

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots$$

over the interval $-\pi \le x \le \pi$ with weight $r(x) = 1$. It is left as a simple exercise to verify that these functions are the eigenfunctions of the Sturm–Liouville problem

$$y'' + \lambda y = 0, \quad y(-\pi) = y(\pi) = 0.$$

The **Fourier series** for a function $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

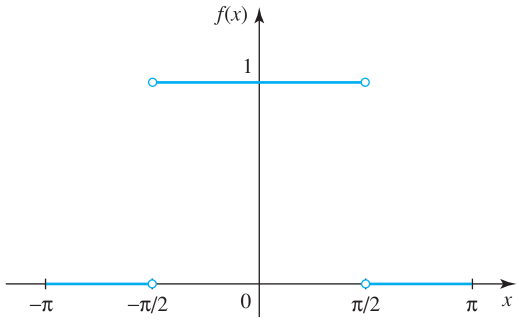


FIGURE 8.13 The rectangular pulse.

where from (132), the **Fourier coefficients** are

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \quad n = 1, 2, \ldots.$$

The formulas for the a_n and b_n are called the **Euler formulas** for the Fourier coefficients.

In anticipation of Chapter 9, let us use these results to find the Fourier series of the (discontinuous) rectangular pulse function

$$f(x) = \begin{cases} 0, & -\pi < x < -\pi/2 \\ 1, & -\pi/2 < x < \pi/2 \\ 0, & \pi/2 < x < \pi \end{cases}$$

shown in Fig. 8.13.

The discontinuities in $f(x)$ cause no problem when deriving the coefficients a_n and b_n because integrals of finite discontinuous functions are well defined:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 1\,dx = \frac{1}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos nx\,dx = \frac{2}{n\pi}\sin\left(\frac{1}{2}n\pi\right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \pm\frac{2}{n\pi} & \text{if } n \text{ is odd.} \end{cases}$$

A similar calculation shows that

$$b_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\sin nx\,dx = \left[\frac{-1}{n\pi}\cos nx\right]_{-\pi/2}^{\pi/2} = 0, \quad n = 1, 2, \ldots.$$

Substituting for the coefficients in the Fourier series gives

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(2n-1)x}{2n-1},$$

and so

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \cdots\right), \quad -\pi \le x \le \pi.$$

Notice that although $f(x)$ is discontinuous at $x = \pm\pi/2$, the Fourier series is defined at these points and has the value 1/2. ■

This example illustrates the fact that a Fourier series expansion (and indeed any eigenfunction expansion) of $f(x)$ is defined for all x in its interval of definition, including points where $f(x)$ is discontinuous, or not even defined. Because of this it is necessary to question the use of the equality sign in (129) and to reinterpret its meaning at points of discontinuity of $f(x)$. More will be said about this in Chapter 9 in connection with Fourier series.

Some comments will be offered later about the convergence of eigenfunction expansions in general, and their behavior at points of discontinuity of $f(x)$ when the completeness of sets of orthogonal functions is discussed.

EXAMPLE 8.28

a Fourier–Legendre expansion

A Fourier–Legendre expansion The expansion of a function $f(x)$ in terms of Legendre polynomials $P_n(x)$ over the interval $-1 \le x \le 1$ is called a **Fourier–Legendre expansion**, and it takes the form

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) = a_0 + a_1 P_1(x) + \cdots. \tag{133}$$

From (135) the coefficients a_n are determined by

$$a_n = \frac{\int_a^b r(x)\varphi_n(x) f(x)dx}{\|\varphi_n(x)\|^2} = \left[\frac{2n+1}{2}\right] \int_{-1}^{1} f(x)P_n(x)dx, \quad n = 0, 1, \ldots.$$

As any polynomial of degree m can be expressed as a linear combination of $P_0(x), P_1(x), \ldots, P_m(x)$, it follows from the orthogonality condition that

$$\int_{-1}^{1} x^m P_n(x)dx = 0 \text{ for } n > m.$$

The Fourier–Legendre expansion of the discontinuous function

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$$

is determined as follows. From (133),

$$a_n = \left(\frac{2n+1}{2}\right) \int_{-1}^{1} f(x)P_n(x)dx = \left(\frac{2n+1}{2}\right) \int_{0}^{1} P_n(x)dx. \tag{134}$$

If we substitute for $P_n(x)$, it then follows that the first few coefficients in the expansion are

$$a_0 = \frac{1}{2}, a_1 = \frac{3}{4}, a_2 = 0, a_3 = -\frac{7}{16}, \ldots,$$

so the required expansion is

$$f(x) = \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \cdots.$$

Here also this Fourier-Legendre expansion attributes a value to $f(x)$ at its point of discontinuity at $x = 0$, and a closer examination shows that the value determined by the expansion is 1/2. ■

EXAMPLE 8.29

a Fourier–Bessel expansion

Fourier–Bessel expansions A function $f(x)$ can be expanded over the interval $0 \le x \le a$ in terms of the Bessel function J_n, with n fixed, to obtain a **Fourier–Bessel** expansion of the form

$$f(x) = \sum_{r=1}^{\infty} a_r J_n(j_{n,r}x/a) = a_1 J_n(j_{n,1}x/a) + a_2 J_n(j_{n,2}x/a) + \cdots, \tag{135}$$

where

$$a_r = \left(\frac{2}{a^2}\right) \frac{\int_0^a J_n(j_{n,r}x/a) f(x)dx}{[J_{n+1}(j_{n,r})]^2} \tag{136}$$

An expansion of this type will be used in Chapter 18 when solving the oscillations of a circular membrane, such as the membrane covering a circular drum head. ■

EXAMPLE 8.30

a Fourier–Chebyshev expansion

Fourier–Chebyshev expansions The **Fourier–Chebyshev expansion** of a function $f(x)$ over the interval $-1 \le x \le 1$ takes the form

$$f(x) = \sum_{n=0}^{\infty} a_n T_n(x) = a_0 T_0(x) + a_1 T_1(x) + \cdots, \tag{137}$$

where

$$a_n = \frac{\int_{-1}^{1} \frac{f(x)T_n(x)}{\sqrt{1-x^2}} dx}{\|T_n(x)\|^2}, \tag{138}$$

with

$$\|T_0(x)\|^2 = \pi \quad \text{and} \quad \|T_n(x)\|^2 = \frac{1}{2}\pi, \quad n = 1, 2, \ldots.$$

Any polynomial of degree m can be expressed as a linear combination of $T_0(x)$, $T_1(x), \ldots, T_m(x)$, so it follows from the orthogonality conditions that

$$\int_{-1}^{1} \frac{x^m T_n(x)}{\sqrt{1-x^2}} dx = 0 \quad \text{for } n > m.$$ ■

It is now necessary to comment on the interpretation of the equality sign in (129) at points where $f(x)$ is discontinuous. For expansions in terms of orthogonal functions to be useful, they must be able to represent the class of functions that occur in practical applications. This means that an orthogonal set of functions defined over an interval $a \le x \le b$ must always be able to be used to expand functions that are piecewise continuous and differentiable at all but a finite number of points in the interval. For conciseness we will denote this set of functions by PC. In addition, the set of orthogonal functions must be sufficiently rich in functions that there is no function of practical importance that cannot be expanded in this manner.

completeness and convergence

Orthogonal (and orthonormal) sets of functions that have this property are said to be **complete**, and the ones introduced so far can all be shown to have this property of completeness. As sets of orthogonal functions are required to expand both continuous and piecewise continuous functions that belong to class PC, the convergence of these expansions must of necessity be more general in nature than ordinary convergence. It is this more general form of convergence, which will be introduced shortly, that will permit the equality sign in (129) to be interpreted in a special sense at points where $f(x)$ is discontinuous.

The special type of convergence we now introduce is called **convergence in the norm**, **mean-square convergence**, or $\mathbf{L}^2$ **convergence**. This form of convergence is defined by requiring that if a sequence of functions $f_1(x), f_2(x), \ldots$ *converges in the mean* to a function $f(x)$, then

$$\lim_{n\to\infty} \| f_n(x) - f(x)\| = 0, \tag{139}$$

or, more explicitly,

$$\lim_{n\to\infty} \int_a^b r(x)[f_n(x) - f(x)]^2\,dx = 0. \tag{140}$$

When interpreting (139) as (140) it is convenient to omit the square root in the definition of the norm, as this simplifies analysis and does not influence the limit.

The sequence of functions $f_n(x)$ in this definition can be taken to be the nth partial sum of the eigenfunction expansion (129),

$$f_n(x) = \sum_{m=0}^{n} a_m\varphi_m(x) = a_0\varphi_0(x) + a_1\varphi_1(x) + \cdots, \tag{141}$$

where from now on we will assume $\varphi_0(x), \varphi_1(x), \ldots$ to be an orthonormal set of functions so that $\|\varphi_n(x)\|^2 = 1, n = 0, 1, \ldots$. Such an orthonormal set of functions will be complete with respect to the functions $f(x)$ in C if every function in PC can be approximated by (141). Convergence in the norm and ordinary convergence are the same everywhere a function is continuous and differentiable.

We now state without proof the fundamental eigenfunction expansion theorem.

THEOREM 8.4

a fundamental eigenfunction expansion theorem

Eigenfunction expansion theorem Let $f(x)$ and $f'(x)$ have at most a finite number of jump discontinuities in the interval $a \le x \le b$. Then the eigenfunction expansion (129) converges in the mean to $f(x)$ at every point of continuity of $f(x)$ inside this interval, and to the value $\frac{1}{2}[f(c-) + f(c+)]$ at any point c where $f(x)$ is discontinuous. ■

This convergence property has already been demonstrated in Example 8.27, where the Fourier series converged to the value 1/2 at the points where the function was discontinuous. Figure 8.14 shows the result in the general case.

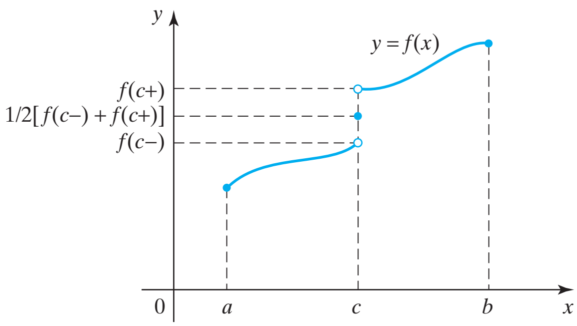


FIGURE 8.14 Convergence of an eigenfunction expansion at a point of discontinuity.

To develop the concept of completeness a little further, we substitute (129) into (140) to obtain

$$\int_a^b r(x)[f_n(x) - f(x)]^2 dx = \int_a^b r(x)[f_n(x)]^2 dx - 2\int_a^b r(x)f(x)f_n(x)dx$$
$$+ \int_a^b r(x)[f(x)]^2 dx = \int_a^b r(x)\left[\sum_{s=0}^{n} a_s\varphi_s(x)\right]^2 dx$$
$$- 2\sum_{s=0}^{n} a_s \int_a^b r(x)f(x)\varphi_s(x)dx + \int_a^b r(x)[f(x)]^2 dx.$$

The orthogonality property of the set of eigenfunctions $\varphi_s(x)$ reduces the first integral on the right to $\sum_{s=0}^{n} a_s^2$, while the definition of a_s shows that the second term on the right can be written $-2\sum_{s=0}^{n} a_s^2$, so the result becomes

$$\int_a^b r(x)[f_n(x) - f(x)]^2 dx = -\sum_{s=0}^{n} a_s^2 + \int_a^b r(x)[f(x)]^2 dx.$$

The integrands of both integrals are nonnegative, and the integral on the right is $\| f(x)\|^2$, so we have established the inequality

$$\sum_{s=0}^{n} a_s^2 \le \int_a^b r(x)[f(x)]^2 dx = \| f(x)\|^2 \quad \text{for all } n \ge 0. \tag{142}$$

Bessel's inequality

This result is called **Bessel's inequality**, and it shows that the sum $\sum_{s=0}^{n} a_s^2$ has the upper bound $\| f(x)\|^2$ as $n \to \infty$. As the terms of the series are nonnegative, the series increases as n increases, so it follows that $\sum_{s=0}^{n} a_s^2$ converges as $n \to \infty$.

If the system of orthonormal functions $\varphi_s(x)$ is complete, result (139) must be true for every function $f(x)$ in the class PC, so that then $\lim_{n\to\infty} \sum_{s=0}^{n} a_s^2 = \| f(x)\|^2$. Consequently, for complete orthonormal systems of functions

$$\sum_{s=0}^{\infty} a_s^2 = \| f(x)\|^2 = \int_a^b r(x)[f(x)]^2 dx. \tag{143}$$

Parseval relation

This result is called the **Parseval relation**.

THEOREM 8.5

Completeness of orthonormal systems Let $\varphi_0(x), \varphi_1(x), \ldots$ be a complete orthonormal set of functions with respect to the set C to which the functions $f(x)$ belong. Then the only continuous function in C that is orthogonal to every function $\varphi_n(x)$ is the zero function $f(x) \equiv 0$. Furthermore, if the restriction of continuity is removed, the only functions that can be orthogonal to every function in the orthonormal set are those with zero norm.

Proof In the first case the vanishing of the norm of $f(x)$ implies that $f(x) \equiv 0$. In the second case, the orthogonality of a function with respect to every eigenfunction implies that the function must be degenerate, and although not identically zero, must have a zero norm. ■

See Chapters 2 and 5 of reference [3.7] for information about eigenfunction expansions and orthonormal sets of functions.

Summary

Eigenfunction expansions have been introduced, and the most important sets of orthogonal functions summarized together with their intervals of definition, weight functions, and orthogonality relationships. Mean-square convergence has been defined and the fundamental eigenfunction theorem stated, and the notion of completeness of systems of orthogonal functions has been explained and related to the Parseval relation.

Appendix 1 (Proofs of Theorem 8.3)

The study of Sturm–Liouville problems is made more concise by the introduction of the notion of a **differential operator *L*** defined as

$$L \equiv \frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x), \tag{144}$$

with the understanding that if y is a suitably differentiable function,

$$L[y] \equiv \frac{d}{dx}\left[p(x)\frac{d}{dx}\right]y(x) + q(x)y(x). \tag{145}$$

Differential operators, of which L is a special case, have the property that when they operate on a function y they produce another function $L[y]$. For example, if

$$L \equiv \frac{d}{dx}\left[x\frac{d}{dx}\right] + 2,$$

and $y(x) = e^{-x}$, then

$$L[e^{-x}] = \frac{d}{dx}\left[x\frac{d[e^{-x}]}{dx}\right] + 2e^{-x} = \frac{d}{dx}[-xe^{-x}] + 2e^{-x} = (1+x)e^{-x}.$$

In terms of the differential operator L in (144), the Sturm–Liouville equation (122) with eigenvalue λ and corresponding eigenfunction φ becomes

$$L[\varphi] + \lambda r\varphi = 0, \tag{146}$$

where φ satisfies suitable boundary conditions.

The proof of the results of Theorem 8.3 that can be given here is simplified by appeal to the following theorem, which is important in its own right.

THEOREM 8.6 **One-dimensional form of Green's theorem** Let L be the linear operator

$$L \equiv \frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x),$$

and, let u, v be any two twice differentiable functions defined on the interval $a \le x \le b$. Then,

(i)

$$\int_a^b uL[v]dx = [p(x)u(x)v'(x)]_a^b - \int_a^b pu'v'\,dx + \int_a^b quv\,dx$$

and

(ii)

$$\int_a^b \{uL[v] - vL[u]\}dx = [p(x)\{u(x)v'(x) - v(x)u'(x)\}]_a^b,$$

called the **Lagrange identity**. Furthermore, if u and v satisfy the boundary conditions

$$A_1\phi(a) + A_2\phi'(a) = 0 \quad \text{and} \quad B_1\phi(b) + B_2\phi'(b) = 0,$$

where ϕ may be either u or v, then

(iii)

$$\int_a^b \{uL[v] - vL[u]\}dx = 0.$$

Proof Result (i) is the one-dimensional form of **Green's first theorem**, and result (ii) is the one-dimensional form of **Green's second theorem**. The three-dimensional forms of these theorems are derived in Chapter 12, Section 12.2. Result (iii) is the consequence of Green's second theorem when u and v satisfy the stated boundary conditions at the ends of the interval $a \le x \le b$.

The proof proceeds as follows. Differentiation of the product $u(pv')$ gives

$$[u(pv')]' = u(pv')' + u'(pv'),$$

so

$$u(pv')' = [puv']' - pu'v'.$$

Recalling the definition of L, we can write

$$uL[v] = [puv']' - pu'v' + quv,$$

so integrating over the interval $a \le x \le b$ gives

$$\int_a^b uL[v]dx = [p(x)u(x)v'(x)]_a^b - \int_a^b pu'v'dx + \int_a^b quvdx,$$

which is result (i).

Result (ii) follows if we interchange u and v in (i) and subtract the result from (i) to obtain

$$\int_a^b \{uL[v] - vL[u]\}dx = [p(x)\{u(x)v'(x) - v(x)u'(x)\}]_a^b.$$

Result (iii) follows from (ii) if we notice that, provided $A_2 \neq 0$, it follows from the boundary conditions at $x = a$ that

$$u'(a) = -(A_1/A_2)u(a) \quad \text{and} \quad v'(a) = -(A_1/A_2)v(a),$$

so

$$[p(uv' - vu')]_{x=a} = -(A_1/A_2)p(a)u(a)v(a) + (A_1/A_2)p(a)u(a)v(a) = 0,$$

and a similar argument shows that, provided $B_2 \neq 0$,

$$[p(uv' - vu')]_{x=b} = 0.$$

Thus, $[p(uv' - vu')]_a^b = 0$, reducing result (ii) to

$$\int_a^b \{uL[v] - vL[u]\}dx = 0,$$

which is result (iii).

Result (iii) is obviously true if the boundary conditions simplify to

$$\phi(a) = 0 \text{ and } \phi(b) = 0 \quad \text{or to } \phi'(a) = 0 \quad \text{and} \quad \phi'(b) = 0,$$

and the modification to the proof needed to show that the result remains true if A_2 and/or B_2 is zero is left as an exercise. ■

JOSEPH LOUIS LAGRANGE (1736–1813)
Lagrange was born in Turin of French extraction and after working in Berlin for twenty years moved to Paris. His many fundamental contributions to mathematics have led to his being regarded as one of the most outstanding mathematicians of his time. He made contributions to algebra, calculus, differential equations, the calculus of variations, and also to mechanics.

We now prove the results in Theorem 8.3 that are straightforward, and refer to the references at the end of the chapter for details of the way in which the more complicated results can be established.

Property 1. The proof of this property is difficult and so will be omitted, but Examples 8.18 to 8.21 illustrate the existence of an ordered sequence of eigenvalues in specific cases.

Property 2. In a regular Sturm–Liouville problem suppose, if possible, that φ and ψ are eigenfunctions corresponding to the single eigenvalue λ. Then each of these functions satisfies the Sturm–Liouville equation, while φ and ψ both satisfy the boundary conditions at $x = a$ so that

$$A_1\varphi(a) + A_2\varphi'(a) = 0 \text{ and } A_1\psi(a) + A_2\psi'(a) = 0.$$

This pair of equations can be considered to determine A_1 and A_2 in terms of φ and ψ at $x = a$. The equations are homogeneous, so there can only be a nontrivial solution for A_1 and A_2 if the determinant of coefficients $W = \varphi(a)\psi'(a) - \varphi'(a)\psi(a)$ vanishes, but this determinant is the Wronskian of the solutions and can only vanish if φ is proportional to ψ, so the result is established.

Property 3. Let φ_m and φ_n be eigenfunctions corresponding to the two distinct eigenvalues λ_m and λ_n of the Sturm–Liouville problem

$$L[y] + \lambda r y = 0$$

defined on $a \le x \le b$ and satisfying homogeneous boundary conditions of the type given in (127). Then it follows that

$$L[\varphi_m] + \lambda_m r\varphi_m = 0 \text{ and } L[\varphi_n] + \lambda_n r\varphi_n = 0.$$

Multiplying the first equation by φ_n and the second by φ_m, subtracting the results, and integrating over the interval $a \le x \le b$ gives

$$\int_a^b \{\varphi_m L[\varphi_n] - \varphi_n L[\varphi_m]\}dx + (\lambda_n - \lambda_m)\int_a^b r\varphi_m\varphi_n\, dx = 0.$$

The first integral vanishes because of the result of Theorem 8.4 (iii), so

$$(\lambda_n - \lambda_m)\int_a^b r\varphi_m\varphi_n dx = 0.$$

The result now follows because $\lambda_n \neq \lambda_m$.

Property 4. The proof is by contradiction. Suppose, if possible, that $\lambda = \alpha + i\beta$ is a complex eigenvalue associated with the complex eigenfunction $\Phi = \varphi + i\psi$. Then as Φ and λ satisfy the Sturm–Liouville equation, we have

$$[p(\varphi + i\psi)']' + [q + (\alpha + i\beta)r](\varphi + i\psi) = 0.$$

This can be written

$$[p\varphi']' + q\varphi + \alpha\varphi r - \beta\psi r + i\{[p\psi']' + q\psi + \beta\varphi r + \alpha\psi r\} = 0.$$

For this to be true, both real and imaginary parts of the equation must vanish, so

$$[p\varphi']' + q\varphi + \alpha\varphi r - \beta\psi r = 0 \quad \text{and} \quad [p\psi']' + q\psi + \beta\varphi r + \alpha\psi r = 0.$$

Multiplying the second equation by i, subtracting it from the first equation, and collecting terms gives

$$[p(\varphi - i\psi)']' + [q + (\alpha - i\beta)r](\varphi - i\psi) = 0,$$

showing that $\overline{\Phi} = \varphi - i\psi$ is an eigenfunction and $\overline{\lambda} = \alpha - i\beta$ is an eigenvalue. As Φ and $\overline{\Phi}$ are linearly independent eigenfunctions, it follows from Theorem 8.3 (3) that

$$\int_a^b r\Phi\overline{\Phi}dx = \int_a^b r(\varphi^2 + \psi^2)dx = 0,$$

but this is impossible because by hypothesis $r(x) \geq 0$ and $\varphi^2 + \psi^2 > 0$. Consequently the assumption that an eigenvalue can be complex is false.

Property 5. Let λ_n be an eigenvalue and φ_n be the corresponding eigenfunction of the Sturm–Liouville equation

$$L[\varphi_n] + \lambda_n r\varphi_n = 0.$$

Multiplication of this equation by φ_n, followed by integration over the interval $a \leq x \leq b$, gives

$$\int_a^b \varphi_n L[\varphi_n]dx + \lambda_n\int_a^b r\varphi_n^2 dx = 0.$$

An application of Theorem 8.4 (i) with $u = v = \varphi_n$ then gives the result

$$\lambda_n = \frac{-[p\varphi_n\varphi_n']_a^b + \int_a^b p(\varphi_n')^2 dx - \int_a^b r\varphi_n^2 dx}{\int_a^b r\varphi_n^2 dx}.$$

Property 6. This follows directly from Property 5 when $q(x) < 0$ and the condition $[p\varphi_n\varphi_n']_a^b \leq 0$ is satisfied.

Property 7. We offer no proof of this result, though as already remarked it is well illustrated by graphs of the Legendre polynomials shown in Fig. 8.1.

Property 8. This follows directly from Property 5 when the stated conditions are imposed, because increasing $p(x)$ will increase the numerator while leaving all other terms unchanged.

Property 9. No proof of this result is offered because it follows from the form of argument used to establish the upper bound property of the Rayleigh quotient given in (128).

Appendix 2 (Norm of $J_n(x)$)

The square of the norm of the Bessel function $J_n(j_{n,r}x/a)$ is the definite integral

$$\|J_n(j_{n,r}x/a)\|^2 = \int_0^a x[J_n(j_{n,r}x/a)]^2 dx = \frac{1}{2}a^2[J_{n+1}(j_{n,r})]^2,$$

and so the norm is

$$\boxed{\|J_n(j_{n,r}x/a)\| = \frac{1}{\sqrt{2}}a[J_{n+1}(j_{n,r})].} \tag{147}$$

This result is most easily derived by considering the case $a = 1$, and then changing variables to obtain the foregoing more general result. Accordingly, we consider the two Bessel equations in Sturm–Liouville form,

$$[xu']' + \left(j_{n,r}^2 x - n^2/x\right)u = 0 \quad \text{and} \quad [xv']' + (k^2 x - n^2/x)v = 0,$$

defined on the interval $0 \le x \le 1$ with bounded solutions that satisfy the boundary conditions $u(1) = v(1) = 0$. These equations have the respective solutions $u(x) = J_n(j_{n,r}x)$ and $v(x) = J_n(kx)$.

Multiplying the first equation by u, the second by v, subtracting the second equation from the first, and integrating over the interval $0 \le x \le 1$ gives, after using Theorem 8.6 (ii) and the result $u'(x) = j_{n,r}J'(j_{n,r}x)$,

$$\int_0^1 xJ_n(j_{n,r}x)J_n(kx)dx = \frac{j_{n,r}J_n(k)J_n'(j_{n,r})}{k^2 - j_{n,r}^2}.$$

We now write this result as

$$\int_0^1 xJ_n(j_{n,r}x)J_n(kx)dx = \left(\frac{j_{n,r}}{k + j_{n,r}}\right)\left(\frac{J_n(k) - J_n(j_{n,r})}{k - j_{n,r}}\right)J_n'(j_{n,r}),$$

where the subtraction of $J_n(j_{n,r})$ in the bracketed term in the numerator leaves the result unchanged because $J_n(j_{n,r}) = 0$.

Taking the limit as $k \to j_{n,r}$, reduces this result to

$$\int_0^1 x[J_n(j_{n,r}x)]^2 dx = \frac{1}{2}[J_n'(j_{n,r})]^2, \quad r = 1, 2, \ldots.$$

It is inconvenient to work with $J_n'(j_{n,r})$, so we relate J_n to J_{n+1} by using recurrence relation (65)′:

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x).$$

Setting $x = j_{n,r}$ causes this to simplify to $J_n'(j_{n,r}) = -J_{n+1}(j_{n,r})$, and so

$$\int_0^1 x[J_n(j_{n,r}x)]^2 dx = \frac{1}{2}[J_{n+1}(j_{n,r})]^2.$$

The more general result follows by making the change of variable $x = z/a$ and then replacing z by x.

EXERCISES 8.11

In Exercises 1 through 3 expand the given polynomials in terms of Legendre polynomials.

1. $4x^3 - 2x^2 + 1$.

2. $3x^3 + x^2 - 4x$.

3. $x^4 + 3x^2 + 2x$.

4. Represent x^2, x^3, and x^4 in terms of Legendre polynomials.

In Exercises 5 through 8 find the first four terms of the Fourier–Legendre expansions of the given functions. In each case graph the four term approximation to $f(x)$ and compare it with the graph of $f(x)$.

5. $f(x) = \begin{cases} 1, & -1 \le x \le 0 \\ x, & 0 < x \le 1. \end{cases}$

6. $f(x) = \begin{cases} 1 + x, & -1 \le x \le 0 \\ 1 - x, & 0 < x \le 1. \end{cases}$

7. $f(x) = \begin{cases} 0, & -1 \le x < -1/2 \\ 1, & -1/2 < x < 1/2 \\ 1/2, & 1/2 < x < 1. \end{cases}$

8. $f(x) = \begin{cases} -2x, & -1 \le x < 0 \\ x, & 0 \le x \le 1. \end{cases}$

9. Find the first four terms in the Fourier–Legendre expansion of e^x.

10. Find the first four terms in the Fourier–Legendre expansion of e^{-x}.

In Exercises 11 through 13 expand the given polynomials in terms of Chebyshev polynomials.

11. $3x^4 - 4x^2 - x$.

12. $4x^3 + x^2 - 3x + 1$.

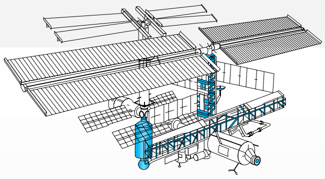
13. $2x^4 - x^3 + x + 3$.

14. Represent x^2, x^3, and x^4 in terms of Chebyshev polynomials.

In Exercises 15 and 16 find the first four terms in the Fourier–Chebyshev expansion of the given function. In each case graph the four term approximation to $f(x)$ and compare it with the graph of $f(x)$.

15. $f(x) = \begin{cases} 2 + x, & -1 < x < 0 \\ 3, & 0 < x < 1. \end{cases}$

16. $f(x) = \begin{cases} -1, & -1 < x < 0 \\ 2x - 1, & 0 < x < 1. \end{cases}$



CHAPTER 8
TECHNOLOGY PROJECTS

Project 1

The Asymptotic Formulas for $J_n(x)$ and $Y_n(x)$

The purpose of this project is to compare plots of the Bessel functions $J_n(x)$ and $Y_n(x)$ with the results obtained from the asymptotic formulas

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{1}{2}n\pi + \frac{1}{4}\pi\right) \quad \text{and}$$

$$Y_n(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{1}{4}\pi(2n+1)\right).$$

Make combined plots of $J_n(x)$ and its asymptotic form, and $Y_n(x)$ and its asymptotic form for $0 \le x \le 30$ for different values of n to illustrate the speed with which the asymptotic approximation tends to the function itself.

Project 2

Chebyshev Approximation

The purpose of this project is to make Chebyshev polynomial approximations of different orders to an asymmetric function $f(x)$ to illustrate the rapidity with which they converge to $f(x)$.

1. Let $f(x) = \sin(5x)(1 + x^2)^{1/4}$ for $-1 \le x \le 1$. Approximate $f(x)$ in terms of the Chebyshev polynomials $T_n(x)$ by the function $f_N(x)$:

$$f_N(x) = \sum_{n=0}^{N} a_n T_n(x).$$

Find the coefficients a_n numerically and make simultaneous plots of $f(x)$ and $f_N(x)$ for $N = 3$, 5, and 7 to show the convergence of $f_N(x)$ to $f(x)$ as N increases.

2. Repeat the calculations with a discontinuous function of your own choice and comment on the behavior of the approximation at the point of discontinuity in the cases when $N = 5, 10, 15, 20, 25$, and 30. Compare your observations with the remarks about the occurrence of the *Gibbs phenomenon* in Fourier series in Chapter 9.

Project 3

Legendre Approximation

The purpose of this project is to make Legendre polynomial approximations of different orders to the function $f(x)$ in Project 2 to illustrate the rapidity with which they converge to $f(x)$.

1. Let $f(x) = \sin(5x)(1 + x^2)^{1/4}$ for $-1 \le x \le 1$. Approximate $f(x)$ in terms of the Legendre polynomials $P_n(x)$ by the function $f_N(x)$:

$$f_N(x) = \sum_{n=0}^{N} a_n P_n(x).$$

Find the coefficients a_n numerically and make simultaneous plots of $f(x)$ and $f_N(x)$ for $N = 3$, 5, and 7 to show the convergence of $f_N(x)$ to $f(x)$ as N increases.

2. Repeat the calculations with a discontinuous function of your own choice and comment on the behavior of the approximation at the point of discontinuity for the cases $N = 5, 10, 15, 20, 25$, and 30. Compare your observations with the remarks about the occurrence of the oscillatory behavior of approximations near a finite jump discontinuity described in Chapter 9 on Fourier series, where the effect is called the *Gibbs phenomenon*.

Project 4

Bessel Function Approximation

The purpose of this project is to make Bessel function approximations of different orders to a function $f(x)$ over a given interval to illustrate the rapidity with which they converge to $f(x)$.

1. Approximate $f(x) = (1 + x^3)\sin x$ over the interval $0 \le x \le \pi$ in terms of the Bessel function $J_1(x)$ by the function $f_N(x)$

$$f_N(x) = \sum_{r=1}^{N} a_r J_1(j_{1,r}x/\pi),$$

where $j_{1,r}$ is the rth zero of $J_1(x)$ listed in Table 8.1. Find the coefficients a_n numerically and make simultaneous plots of $f(x)$ and $f_N(x)$ for $N = 3$, 5, and 7 to show the convergence of $f_N(x)$ to $f(x)$ as N increases.

2. Repeat the calculation with a continuous function $f(x)$ of your own choice. When making the series expansion in terms of the Bessel function $J_n(x)$, use the value $n = 0$ if $f(0) \neq 0$ and $n = 1$ if $f(0) = 0$.

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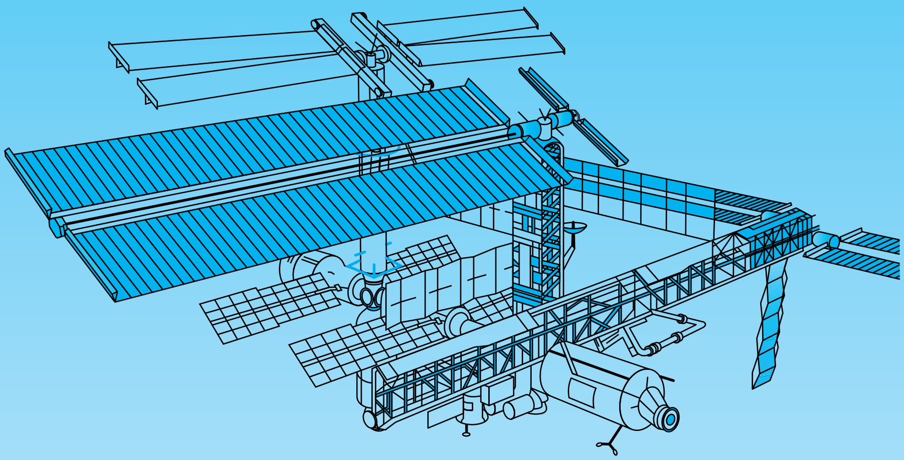
PART FOUR

FOURIER SERIES, INTEGRALS, AND THE FOURIER TRANSFORM



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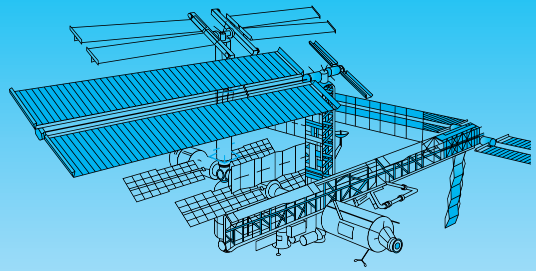
CHAPTER 9



Fourier Series

When analyzing situations as diverse as electrical oscillations, vibrating mechanical systems, longitudinal oscillations in crystals, and many other physical phenomena, Fourier series are found to arise naturally. Furthermore, the individual terms in a Fourier series often have an important physical interpretation. In a vibrating mechanical system, for example, each component of a Fourier series representation of the overall vibration represents a fundamental mode of vibration. The full Fourier series shows how each mode contributes to the solution, and which are the most significant modes. This information can often be used to advantage, either by showing how the modes can be utilized to achieve a desired effect, or by using the information to enable systems to be constructed that minimize undesirable vibrations. It is for these and other reasons that it is necessary for engineers and physicists to study the properties of Fourier series.

9.1 Introduction to Fourier Series

A **Fourier series** representation of a function $f(x)$ over the interval $-\pi \leq x \leq \pi$ is an expression of the form

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$
$$= a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots, \qquad (1)$$

where the coefficients $a_0, a_1, \ldots, b_1, b_2, \ldots$ are determined by the function $f(x)$.

It is important to notice that the Fourier series representation of $f(x)$ contains two infinite sums, one of even functions (the cosines) and the other of odd functions (the sines). It will be recalled that a function $f(x)$ defined in the interval $-L \leq x \leq L$ is said to be an **even function** in the interval if

even and odd function

$$f(-x) = f(x), \qquad (2)$$

and to be an **odd function** in the interval if

$$f(-x) = -f(x). \tag{3}$$

The cosine function is an even function because $\cos(-x) = \cos x$ in agreement with the definition in (2). As this is true for all x, the function $\cos x$ is an even function for $-\infty < x < \infty$. Similarly, $\sin x$ is an odd function because $\sin(-x) = -\sin x$ in agreement with the definition in (3). This also is true for all x, so the function $\sin x$ is an odd function for $-\infty < x < \infty$.

Most functions are neither even nor odd, but any function in an interval $-L \leq x \leq L$ can be expressed as the sum of an even function and an odd function defined over the interval. To see why this is, let $f(x)$ be an arbitrary function defined over the interval $-L \leq x \leq L$, and write it in the form

$$f(x) = \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x)) \quad \text{for } -L \leq x \leq L. \tag{4}$$

Then the function

$$h(x) = \frac{1}{2}(f(x) + f(-x)) \tag{5}$$

is seen to be an *even* function, because $h(-x) = h(x)$, whereas the function

$$g(x) = \frac{1}{2}(f(x) - f(-x)) \tag{6}$$

is seen to be an odd function, because $g(-x) = -g(x)$, so the assertion is proved.

EXAMPLE 9.1

Classify the following functions as even, odd, or neither.

(a) $\cosh x$. (b) $\sinh x$. (c) $x^2 + \sin x$. (d) $1 + x^2 + 3x^4$.

Solution (a) As $\cosh(-x) = \cosh x$ for all x, the function $\cosh x$ is an even function for all x. (b) As $\sinh(-x) = -\sinh x$ for all x, the function $\sinh x$ is an odd function for all x. (c) $(-x)^2 = x^2$, so x^2 is an even function for all x, while $\sin x$ is an odd function for all x, so the function $x^2 + \sin x$ is neither even nor odd. In this case the function $x^2 + \sin x$ is already expressed as the sum of an even function and an odd function. (d) Set $f(x) = 1 + x^2 + 3x^4$. Then $f(-x) = 1 + (-x)^2 + (-x)^4 = f(x)$, so $f(x)$ is an even function. This result can be obtained by a different form of argument as follows. A constant does not change when the sign of x is changed, so all constants are even functions and, in particular, 1 is an even function. The function x^2 has already been shown to be an even function, and the function $3x^4$ is an even function because $3(-x)^4 = 3x^4$. Thus, as the function $1 + x^2 + 3x^4$ is a sum of three even functions, it must be an even function. ■

To arrive at a formula for the a_n in (1) corresponding to a given function $f(x)$, result (1) is first multiplied term by term by $\cos nx$ to obtain

deriving formulas for a_n and b_n

$$\begin{aligned} f(x)\cos nx = {} & a_0 \cos nx + a_1 \cos x \cos nx + a_2 \cos 2x \cos nx + a_3 \cos 3x \cos nx \\ & + \cdots + a_{n-1}\cos(n-1)x \cos nx + a_n \cos^2 nx \\ & + a_{n+1}\cos(n+1)x \cos nx + \cdots + b_1 \sin x \cos nx \\ & + b_2 \sin 2x \cos nx + \cdots . \end{aligned}$$

Integrating this result over the interval $-\pi \le x \le \pi$ gives

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos nx dx &= a_0\int_{-\pi}^{\pi}\cos nx dx + a_1\int_{-\pi}^{\pi}\cos x\cos nx dx \\ &\quad + a_2\int_{-\pi}^{\pi}\cos 2x\cos nx dx + a_3\int_{-\pi}^{\pi}\cos 3x\cos nx dx + \cdots \\ &\quad + a_{n-1}\int_{-\pi}^{\pi}\cos(n-1)x\cos nx dx + a_n\int_{-\pi}^{\pi}\cos^2 nx dx \\ &\quad + a_{n+1}\int_{-\pi}^{\pi}\cos(n+1)x\cos nx dx + \cdots + b_1\int_{-\pi}^{\pi}\sin x\cos nx dx \\ &\quad + b_2\int_{-\pi}^{\pi}\sin 2x\cos nx dx + \cdots.\end{aligned}$$

The orthogonality properties of the sine and cosine functions listed in entry 1 of the summary of main sets of orthogonal functions in Section 8.11 shows that all integrals on the right with the exception of the one with the integrand $\cos^2 nx$ vanish, giving rise to the result

$$\int_{-\pi}^{\pi} f(x)\cos nx dx = a_n\int_{-\pi}^{\pi}\cos^2 nx dx.$$

However, $\int_{-\pi}^{\pi}\cos^2 nx dx = \pi$, for $n \ne 0$ and $\int_{-\pi}^{\pi} 1.dx = 2\pi$, so

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \quad \text{and} \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \quad \text{for } n = 1, 2, \ldots.$$

A similar argument involving the multiplication of the Fourier series (1) by $\sin nx$ followed by integration over the interval $-\pi \le x \le \pi$ and use of the orthogonality properties of $\sin nx$ shows the coefficients b_n are given by

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \quad \text{for } n = 1, 2, \ldots.$$

the Euler formulas

These results are the **Euler formulas** for the **Fourier coefficients** a_n and b_n, and for future reference they are now listed, together with the associated Fourier series representation of $f(x)$.

the Fourier series representation

Fourier series representation of $f(x)$ over the interval $-\pi \le x \le \pi$

Let the function $f(x)$ be defined on the interval $-\pi \le x \le \pi$. Then the Fourier coefficients a_n and b_n in the Fourier series representation of $f(x)$

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) \tag{7}$$

are given by the Euler formulas

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \ldots \quad \text{for } n = 1, 2, \ldots. \end{aligned} \tag{8}$$

The arguments used to derive the Euler formulas in (8) are not rigorous, because the term by term integration needs to be justified and the convergence of the Fourier series representation of $f(x)$ to the function $f(x)$ itself has not been examined, so the use of an equality sign in (1) and (7) must be questioned.

JEAN BAPTISTE JOSEPH (BARON) FOURIER (1768–1830)
A remarkable French physicist who was also an outstanding mathematician. He was orphaned at eight, and educated in a military school run by the Benedictines who then gave him a lectureship in mathematics. He later moved to a chair at the Ecole Polytechnique in Paris, and later to Grenoble where he was appointed Prefect by Napoleon. His experiments on heat conduction while in Grenoble, suggested by Newton's Law of Cooling, led him to propose his law of heat conduction (Fourier's Law) and to the publication of his most important *Theorie Analytique de la Chaleur* in which he introduced the representation of arbitrary function over an interval in terms of trigonometric functions, now called Fourier series. He was created a Baron by Napoleon in 1808.

In fact, the preceding approach can be fully justified for all functions $f(x)$ that arise in practical situations, and we will see later that the equality sign can be used wherever $f(x)$ is continuous, whereas at points where $f(x)$ experiences a finite jump discontinuity the value assumed by the Fourier series representation is the average of the values to the immediate left and right of the jump. It is for these reasons that in more advanced accounts the equality sign in (7) is replaced by a tilde $\sim$, because this indicates that a relationship exists between a function $f(x)$ and its Fourier series representation without indicating that it is necessarily a strict equality. When this notation is used, the connection between $f(x)$ and its Fourier series is shown by writing

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{9}$$

fundamental interval, periodicity, and periodic extension

The interval of integration $-\pi \le x \le \pi$ used when deriving the Euler formulas is called the **fundamental interval** of the Fourier series, and the Fourier coefficients will always be defined provided the integral $\int_{-\pi}^{\pi} f(x)dx$ exists. Although Fourier series comprise only even and odd functions, results (4) to (6) allow a Fourier series to represent arbitrary functions that are neither even nor odd.

A Taylor series expansion of a function $f(x)$ about a point x_0 requires the function to be repeatedly differentiable at x_0. However, the coefficients of a Fourier series are defined in terms of definite integrals that are still defined when $f(x)$ has finite jump discontinuities in the fundamental interval, so the Euler formulas still remain valid when $f(x)$ is discontinuous. It is this property of a definite integral that makes a Fourier series representation of a function more general than a Taylor series expansion.

The properties of Fourier series reflect the *periodicity* of the sine and cosine functions used in the expansion, where the *period* of a periodic function is defined as follows. A function $g(x)$ is said to be **periodic** with **period** T if

$$g(x+T) = g(x) \tag{10}$$

for all x, and T is the *smallest* value for which (10) is true. A periodic function $g(x)$ may either be continuous or discontinuous, and an example of a continuous periodic function with period T is shown in Fig. 9.1.

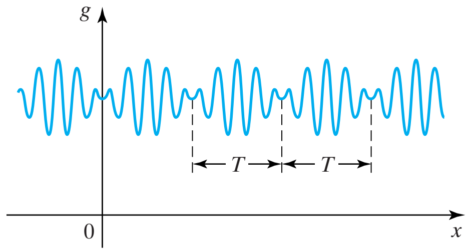


FIGURE 9.1 A continuous periodic function $g(x)$ with period T.

The functions 1, $\cos nx$, and $\sin nx$ in the Fourier series representation (7) of $f(x)$ are all periodic with period 2π, so the *Fourier series representation* of $f(x)$ defined on the interval $-\pi < x < \pi$ is also periodic with period 2π. It does not necessarily follow that outside the fundamental interval the function $f(x)$ coincides with its Fourier series representation, because the behavior of $f(x)$ outside the fundamental interval does not enter into the Euler formulas. Each representation of $f(x)$ in an interval of the form $(2n-1)\pi < x < (2n+1)\pi$, with $n = 0, \pm 1, \pm 2, \ldots$, is called a **periodic extension** of the fundamental interval for $f(x)$.

In Chapter 8, Example 8.22, the discontinuous rectangular pulse function

$$f(x) = \begin{cases} 0, & -\pi < x < -\pi/2 \\ 1, & -\pi/2 < x < \pi/2 \\ 0, & \pi/2 < x < \pi \end{cases}$$

was shown to be represented by the Fourier series

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \frac{\cos 7x}{7} + \cdots\right] \quad \text{for all } x. \tag{11}$$

If this function $f(x)$ is defined for all x by the periodicity condition $f(x + 2\pi) = f(x)$, its graph takes the form shown in Fig. 9.2. Figure 9.3 shows the graph of the first five terms of the Fourier series representation (11) in the fundamental interval.

This simple example emphasizes two important issues that always arise when working with Fourier series representations of functions:

1. The need to interpret the equality sign in (7) at any point $x = x_0$ in the fundamental interval where $f(x)$ is discontinuous.
2. The fact that the Fourier series of a function and the periodic extensions of the function will only coincide when the function $f(x)$ is itself periodic with a period equal to the fundamental interval.

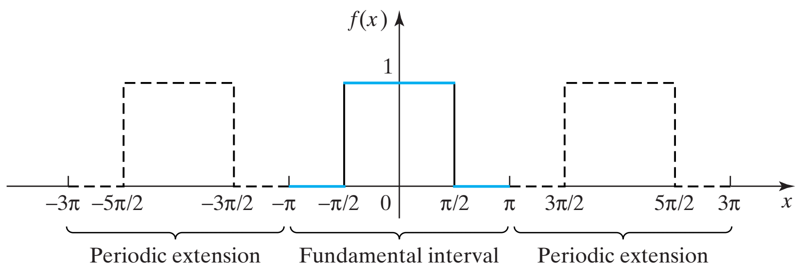


FIGURE 9.2 The periodic rectangular pulse function $f(x)$.

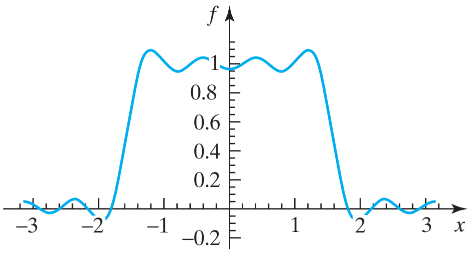


FIGURE 9.3 Graph of the first five terms of the Fourier series of $f(x)$.

An example of the difference that can arise between the behavior of a nonperiodic function $f(x)$ and its periodic extensions is illustrated in Fig. 9.4 in the case of the function

$$f(x) = \begin{cases} 1/2, & x < -\pi \\ 0, & -\pi < x < -\pi/2 \\ 1, & -\pi/2 < x < \pi/2 \\ 0, & \pi/2 < x < \pi \\ 1/4, & x > \pi. \end{cases}$$

The periodic extensions of $f(x)$ in its fundamental interval $-\pi \le x \le \pi$ shown as dashed lines are, of course, the same as those in Fig. 9.2, though in this case the behavior of $f(x)$ outside the fundamental interval is entirely different.

EXAMPLE 9.2

some illustrative examples

Find the Fourier series representation of

$$f(x) = \begin{cases} \sin 2x, & -\pi < x < -\pi/2 \\ 0, & -\pi/2 \le x \le 0 \\ \sin 2x, & 0 < x \le \pi. \end{cases}$$

Solution The function $f(x)$ is continuous over the fundamental interval $-\pi \le x \le \pi$, but it is defined in piecewise manner, so the Fourier coefficients must be determined by integrating the Euler equations (8) in a corresponding manner. We have

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{-\pi/2} \sin 2x dx + \frac{1}{2\pi}\int_{0}^{\pi} \sin 2x dx \\ &= \frac{1}{2\pi}[-(1/2)\cos 2x]_{-\pi}^{-\pi/2} + \frac{1}{2\pi}[-(1/2)\cos 2x]_0^{\pi} = \frac{1}{2\pi} + 0 = \frac{1}{2\pi}. \end{aligned}$$

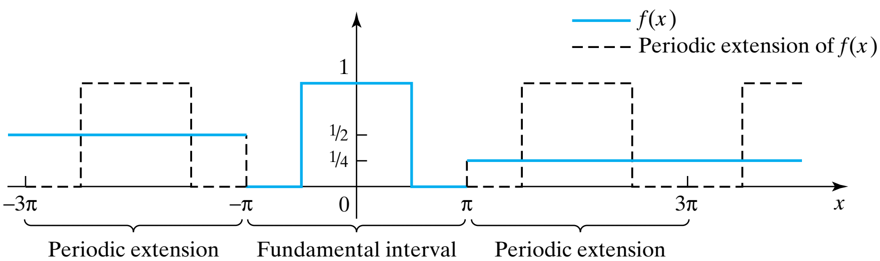


FIGURE 9.4 A nonperiodic function defined for all x, and the periodic extensions of the function in its fundamental interval.

Similarly,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{-\pi/2} \sin 2x \cos nx dx + \frac{1}{\pi}\int_{0}^{\pi} \sin 2x \cos nx dx$$

$$= \frac{-2}{\pi}\left[\frac{\cos n\pi + \cos(n\pi/2)}{n^2-4}\right]_{-\pi}^{-\pi/2} + \frac{2}{\pi}\left[\frac{\cos n\pi - 1}{n^2-4}\right]_0^{\pi}, \quad \text{for } n \neq 2$$

$$= \frac{-2[1+\cos(n\pi/2)]}{\pi(n^2-4)}, \quad \text{for } n \neq 2.$$

As the denominator in the expression for a_n is zero when $n = 2$, in order to find a_2 it is necessary to return to the Euler formula for a_n and set $n = 2$ *before* integrating, when we obtain

$$a_2 = \frac{1}{\pi}\int_{-\pi}^{-\pi/2} \sin 2x \cos 2x dx + \frac{1}{\pi}\int_0^{\pi} \sin 2x \cos 2x dx = 0 + 0 = 0.$$

The Euler formula for b_n becomes

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{-\pi/2} \sin 2x \sin nx dx + \frac{1}{\pi}\int_0^{\pi} \sin 2x \sin nx dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n-2)x}{n-2} - \frac{\sin(n+2)x}{n+2}\right]_{-\pi}^{-\pi/2} + \frac{1}{2\pi}\left[\frac{\sin(n-2)x}{n-2} - \frac{\sin(n+2)x}{n+2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\frac{\sin(n\pi/2)}{(n^2-4)}, \quad \text{for } n \neq 2.$$

As the denominator in the expression for b_n is zero for $n = 2$, to find b_2 we must set $n = 2$ in the Euler formula for b_2 before integrating, as a result of which we find that

$$b_2 = \frac{1}{\pi}\int_{-\pi}^{-\pi/2} \sin^2 2x dx + \frac{1}{\pi}\int_0^{\pi} \sin^2 2x dx$$

$$= \frac{1}{4\pi}[2x - \sin 2x \cos 2x]_{-\pi}^{-\pi/2} + \frac{1}{4\pi}[2x - \sin 2x \cos 2x]_0^{\pi}$$

$$= \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$$

Combining the preceding results shows the first few Fourier coefficients to be

$$a_0 = \frac{1}{2\pi}, \quad a_1 = \frac{2}{3\pi}, \quad a_2 = 0, \quad a_3 = -\frac{2}{5\pi}, \quad a_4 = -\frac{1}{3\pi}, \quad a_5 = -\frac{2}{21\pi},$$

$$b_1 = -\frac{2}{3\pi}, \quad b_2 = \frac{3}{4}, \quad b_3 = -\frac{2}{5\pi}, \quad b_4 = 0, \quad b_5 = \frac{2}{21\pi}, \cdots.$$

When these coefficients are used, the first few terms of the Fourier series for $f(x)$ are seen to be

$$f(x) = \frac{1}{2\pi} + \frac{1}{\pi}\left(\frac{2}{3}\cos x - \frac{2}{5}\cos 3x - \frac{1}{3}\cos 4x - \frac{2}{21}\cos 5x + \cdots\right)$$

$$+ \frac{1}{\pi}\left(-\frac{2}{3}\sin x + \frac{3\pi}{4}\sin 2x - \frac{2}{5}\sin 3x + \frac{2}{21}\sin 5x + \cdots\right).$$

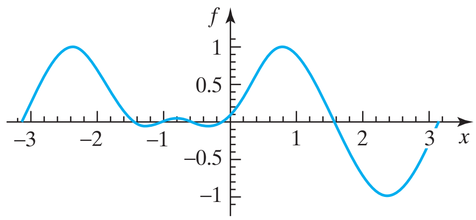


FIGURE 9.5 Fourier series approximation for $f(x)$.

This example illustrates how when a sine function (or a cosine function) with an argument mx with m an integer occurs in a piecewise defined function, its Fourier coefficients a_m and b_m must be found from the Euler formulas with n set equal to m before integration. Figure 9.5 shows a graph of this Fourier series approximation to $f(x)$ up to and including the terms in $\cos 5x$ and $\sin 5x$. ■

It is useful to have a special name for finite approximations to Fourier series such as the one used to construct the graph in Fig. 9.5. Because of this it is usual to call the approximation

$$S_N(x) = a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx) \tag{12}$$

Nth partial sum

to the full Fourier series in (7) the **Nth partial sum** of the Fourier series. Thus, the graph in Fig. 9.5 shows the fifth partial sum $S_5(x)$ of the function $f(x)$ defined in Example 9.2. The Fourier series in (7) is related to its Nth partial sum $S_n(x)$ by the limit

$$f(x) = a_0 + \sum_{n=0}^{\infty}(a_n \cos nx + b_n \sin nx) = \lim_{N\to\infty} S_N(x). \tag{13}$$

Not every function has a Fourier series involving an infinite number of terms, as can be seen by considering the function $f(x) = 1 + 2\sin x \cos x$. When this is rewritten as $f(x) = 1 + \sin 2x$, it is recognized that it is, in fact, its own Fourier series.

There is nothing special about the choice of $-\pi \le x \le \pi$ as a fundamental interval, and it is often necessary to take the fundamental interval to be $-L \le x \le L$. Results (7) and (8) generalize immediately once it is recognized that the set of functions

$$1,\ \cos\frac{\pi x}{L},\ \cos\frac{2\pi x}{L},\ \cos\frac{3\pi x}{L},\ \ldots,\ \sin\frac{\pi x}{L},\ \sin\frac{2\pi x}{L},\ \sin\frac{3\pi x}{L},\ \ldots$$

form an orthogonal set over the interval $-L \le x \le L$. This can be seen by using routine integration to show that

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0 \quad \text{for all integers } m \text{ and } n, \tag{14}$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \begin{cases} 0 & \text{for } m \ne n \\ L & \text{for } m = n \end{cases} \quad \text{for all integers } m \text{ and } n, \tag{15}$$

and

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx = \begin{cases} 0 & \text{for } m \neq n \\ L & \text{for } m = n \neq 0 \text{ for all integers } m \text{ and } n \\ 2L & \text{for } m = n = 0. \end{cases} \tag{16}$$

The Fourier series of a function $f(x)$ defined on the interval $-L \le x \le L$ becomes

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right), \tag{17}$$

and the corresponding Euler formulas for the a_n and b_n follow as before. The coefficients a_n are obtained by multiplying (17) by $\cos\frac{n\pi x}{L}$ and integrating over the interval $-L \le x \le L$, while the b_n follow by multiplying (17) by $\sin\frac{n\pi x}{L}$ and integrating over the same interval. The result is as follows, though the details are left as an exercise.

Fourier series over $-L \le x \le L$

Fourier series representation of $f(x)$ over the interval $-L \le x \le L$

Let the function $f(x)$ be defined on the interval $-L \le x \le L$. Then the Fourier coefficients a_n and b_n in the Fourier series representation of $f(x)$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \tag{18}$$

are given by the Euler formulas

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx,$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad \text{for } n = 1, 2, \ldots. \tag{19}$$

EXAMPLE 9.3

Find the Fourier series representation of $f(x) = x + 1$ for $-1 \le x \le 1$.

Solution In this case $L = 1$, so using integration by parts we find that

$$a_0 = \frac{1}{2}\int_{-1}^{1}(x+1)dx = 1, \quad a_n = \int_{-1}^{1}(x+1)\cos n\pi x\,dx = \frac{\cos n\pi x}{n^2\pi^2} + \frac{x\sin n\pi x}{n\pi} + \left.\frac{\sin n\pi x}{n\pi}\right]_{-1}^{1} = 0$$

and

$$b_n = \int_{-1}^{1}(x+1)\sin n\pi x\,dx = \frac{\sin n\pi x}{n^2\pi^2} - \frac{x\cos n\pi x}{n\pi} - \left.\frac{\cos n\pi x}{n\pi}\right]_{-1}^{1} = \frac{2(-1)^{n+1}}{n\pi},$$

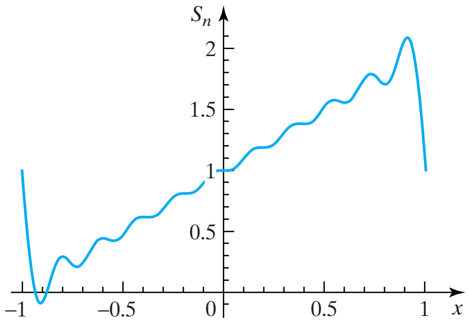


FIGURE 9.6 The partial sum approximation $S_{10}(x)$.

for $n = 1, 2, \ldots$, where we have used the fact that $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$ for n a positive integer. Substituting these coefficients into (18) shows the required Fourier series representation to be

$$f(x) = 1 + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\pi x, \quad \text{for } -1 \le x \le 1.$$

A graph of the partial sum approximation $S_{10}(x)$ to $f(x)$ is shown in Fig. 9.6. ■

As cosines are even functions and sines are odd functions, it is to be expected that a Fourier series representation of an even function will only contain cosine terms, whereas a Fourier series representation of an odd function will only contain sine functions. These properties form the basis of the following result that simplifies the task of finding Fourier series representations of even and odd functions.

expanding even and odd functions

Fourier series of even and odd functions

If $f(x)$ is an even function defined on the interval $-L \le x \le L$, then

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \quad \text{with } a_0 = \frac{1}{L}\int_0^L f(x)dx,$$
$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$

for $n = 1, 2, \ldots$; if $f(x)$ is an odd function, then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \quad \text{with } b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx,$$
$$\text{for } n = 1, 2, \ldots,$$

The justification of these results is as follows. To find the form taken by the Fourier coefficients a_n of an even function, and why its Fourier coefficients b_n vanish, we will consider an even function $f(x)$ defined over the interval $-L \le x \le L$.

By definition,

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{2L}\int_{-L}^{0} f(x)dx + \frac{1}{2L}\int_{0}^{L} f(x)dx.$$

Setting $x = -u$ in the first integral on the right gives

$$\frac{1}{2L}\int_{-L}^{0} f(x)dx = -\frac{1}{2L}\int_{L}^{0} f(-u)du.$$

As f is an even function, $f(-u) = f(u)$, so using this result, changing the sign of the integral by interchanging its limits, and then replacing the dummy variable u by x gives

$$\frac{1}{2L}\int_{-L}^{0} f(x)dx = \frac{1}{2L}\int_{0}^{L} f(x)dx.$$

When this is combined with the original expression for a_0 we find that

$$a_0 = \frac{1}{L}\int_{0}^{L} f(x)dx,$$

and a strictly analogous argument shows that

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos n\pi x dx \quad \text{for } n = 1, 2, \ldots.$$

The Fourier coefficients b_n are given by

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = \frac{1}{L}\int_{-L}^{0} f(x)\sin\frac{n\pi x}{L}dx + \frac{1}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}dx.$$

Setting $x = -u$ in the integral taken over the interval $-L \le x \le 0$ gives

$$\frac{1}{L}\int_{-L}^{0} f(x)\sin\frac{n\pi x}{L}dx = -\frac{1}{L}\int_{L}^{0} f(-u)\sin\left(-\frac{n\pi u}{L}\right)du.$$

We now use the fact that f is an even function, so $f(-u) = f(u)$, together with the fact that the sine function is an odd function. Reversal of the limits coupled with changing the sign and replacing u by x gives

$$\frac{1}{L}\int_{-L}^{0} f(x)\sin\frac{n\pi x}{L}dx = -\frac{1}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}dx.$$

Finally, using this result in the original expression for b_n gives

$$b_n = \frac{1}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}dx - \frac{1}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}dx = 0 \quad \text{for } n = 1, 2, \ldots,$$

and the result is proved.

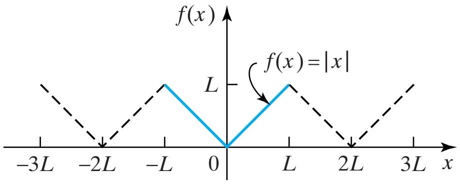


FIGURE 9.7 The function $f(x) = |x|$ in $-L \leq x \leq L$ and two periodic extensions.

A similar argument shows that if $f(x)$ is an odd function over $-L \leq x \leq L$, then

$$a_n = 0 \quad \text{for } n = 0, 1, 2, \ldots,$$

and

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \quad \text{for } n = 1, 2, \ldots,$$

and the results have been established.

EXAMPLE 9.4 Find the Fourier series representation of $f(x) = |x|$ in the interval $-L \leq x \leq L$.

Solution The graph of this even function, together with two of its periodic extensions outside the fundamental interval $-L \leq x \leq L$, is shown in Fig. 9.7.

The Euler formula for the coefficients a_n of the *even* function $|x|$ defined as

$$|x| = \begin{cases} -x & \text{for } < 0 \\ x & \text{for } x \geq 0 \end{cases}$$

gives

$$a_0 = \frac{1}{L}\int_0^L x dx = \frac{L}{2}$$

and

$$a_n = \frac{2}{L}\int_0^L x\cos\frac{n\pi x}{L}dx = \frac{2}{L}\left[\frac{L^2\cos\frac{n\pi x}{L}}{n^2\pi^2} + \frac{Ln\pi x\sin\frac{n\pi x}{L}}{n^2\pi^2}\right]_0^L, \quad \text{for } n = 1, 2, \ldots.$$

a convenient representation of cos $n\pi$

If we use the fact that $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$ when n is a positive integer, it then follows that

$$a_n = \frac{2L}{n^2\pi^2}[(-1)^n - 1] \quad \text{for } n = 1, 2, \ldots,$$

and so

$$a_n = -\frac{4L}{n^2\pi^2} \quad \text{when } n \text{ is odd}$$

and

$$a_n = 0 \quad \text{when } n \neq 0, \text{ is even.}$$

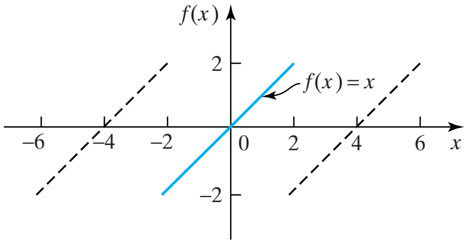


FIGURE 9.8 The function $f(x) = x$ in $-2 \le x \le 2$ and two periodic extensions.

Thus, the Fourier series representation of $f(x) = |x|$ for $-L \le x \le L$ is

$$f(x) = \frac{L}{2} - \frac{4L}{\pi^2}\left(\frac{\cos\dfrac{\pi x}{L}}{1^2} + \frac{\cos\dfrac{3\pi x}{L}}{3^2} + \frac{\cos\dfrac{5\pi x}{L}}{5^2} + \cdots\right).$$

The sequence of positive odd numbers can be written in the form $2n - 1$ with $n = 1, 2, \ldots$, so this last result can be expressed more concisely as

$$f(x) = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos\left(\dfrac{(2n-1)\pi x}{L}\right)}{(2n-1)^2} \quad \text{for } -L \le x \le L. \quad \blacksquare$$

EXAMPLE 9.5 Find the Fourier series representation of $f(x) = x$ on the interval $-2 \le x \le 2$.

Solution A graph of $f(x)$ and two of its periodic extensions outside the fundamental interval $-2 \le x \le 2$ is shown in Fig. 9.8.

Using the fact that $L = 2$, a straightforward calculation gives

$$b_n = \frac{1}{2}\int_{-2}^{2} x \sin\frac{n\pi x}{2}dx = \frac{2}{n^2\pi^2}\left[\sin\frac{n\pi x}{2} - \frac{1}{2}n\pi x\cos\frac{n\pi x}{2}\right]_{-2}^{2}$$

$$= -\frac{4\cos n\pi}{n\pi} = \frac{4(-1)^{n+1}}{n\pi},$$

and as the function is odd all the coefficients $a_n = 0$.

The required Fourier series representation is thus

$$f(x) = \frac{4}{\pi}\left(\frac{\sin\dfrac{\pi x}{2}}{1} - \frac{\sin\pi x}{2} + \frac{\sin\dfrac{3\pi x}{2}}{3} - \cdots\right),$$

which can be written in the more concise form

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{2} \quad \text{for } -2 \le x \le 2. \quad \blacksquare$$

Summary

Fourier series have been defined over more general intervals than $-\pi \le x \le \pi$ and the notion of a periodic extension has been introduced. Attention has been drawn to the behavior of a Fourier series representation at a point of discontinuity of $f(x)$, and the expansion of even and odd functions has been considered.

EXERCISES 9.1

Find the period of each of the functions in Exercises 1 through 6.

1. $\cos x + \sin 2x$.

2. $2\sin 2x - 3\cos \dfrac{x}{3}$.

3. $\sin x \cos x$.

4. $\cos 2x \sin x$.

5. $3\sin \dfrac{x}{3} + \cos \dfrac{x}{2}$.

6. $\cos \dfrac{x}{3} + 5\sin \dfrac{x}{4}$.

In Exercises 7 through 10 (a) sketch the given function in the interval $-3a < x < 3a$, and (b) in the intervals $-3a < x < -a$ and $a < x < 3a$, and state whether the function is periodic.

7. $f(x) = \begin{cases} 0, & x < a/2 \\ 1, & x > a/2. \end{cases}$

8. $f(x) = \begin{cases} -1, & -a < x < 0 \\ 2, & 0 < x < a, \end{cases} \quad f(x+2a) = f(x).$

9. $f(x) = a - |x|$.

10. $f(x) = |\sin \pi x/a|$.

In Exercises 11 and 12 make use of the trigonometric identities $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$ and $\cos(A \pm B) = \cos A\cos B + \sin A\sin B$ to transform the given functions into their (finite) Fourier series.

11. (a) $\sin x \cos x$. (b) $1 - 2\sin^2 x$. (c) $\sin 3x \cos x$.

12. (a) $4\cos 2x \cos 5x$. (b) $\sin x \sin 2x$. (c) $\cos^2 2x - 1/2$.

Verify the following definite integrals that were used when developing a Fourier series representation over the interval $-L < x < L$.

13. $\displaystyle\int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx = 0$ for all integers m and n.

14. $\displaystyle\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx = \begin{cases} 0 & \text{for } m \ne n \\ L & \text{for } m = n, \\ & \text{with } m, n \text{ integers.} \end{cases}$

15. $$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx = \begin{cases} 0 & \text{for } m \ne n \\ L & \text{for } m = n \ne 0 \\ 2L & \text{for } m = n = 0 \text{ for all integers } m \text{ and } n. \end{cases}$$

16. Prove that the product of two even functions and of two odd functions is an even function, and that the product of an even and an odd function is an odd function.

17. Prove that the sum of two even functions is an even function and the sum of two odd functions is an odd function.

18. Prove that if $f(x)$ is an odd function all the Fourier coefficients $a_n = 0$.

19. Evaluate the following integrals that arise when finding the Fourier series expansion of x over the interval $-L < x < L$.

(a) $\displaystyle\int_{-L}^{L} x\sin\frac{\pi x}{L}dx$. (b) $\displaystyle\int_{-L}^{L} x\sin\frac{2\pi x}{L}dx$.

(c) $\displaystyle\int_{-L}^{L} x\sin\frac{3\pi x}{L}dx$.

20. Evaluate the following integrals that arise when finding the Fourier series expansion of x^2 over the interval $-L < x < L$.

(a) $\displaystyle\int_{-L}^{L} x^2\sin\frac{\pi x}{L}dx$. (b) $\displaystyle\int_{-L}^{L} x^2\sin\frac{2\pi x}{L}dx$.

(c) $\displaystyle\int_{-L}^{L} x^2\sin\frac{3\pi x}{L}dx$.

The integrals in Exercises 21 and 22 arise when finding the Fourier series expansion of e^{ax} over the interval $-L < x < L$. Use the result $\cos n\pi = (-1)^n$ for integral values of n to establish the stated result.

21. $\displaystyle\int_{-\pi}^{\pi} e^{ax}\sin nx dx = (-1)^{n+1}\frac{n(e^{a\pi} - e^{-a\pi})}{(a^2+n^2)}$ for integral values of n.

22. $\displaystyle\int_{-\pi}^{\pi} e^{ax}\cos nx dx = (-1)^{n}\frac{a(e^{a\pi} - e^{-a\pi})}{(a^2+n^2)}$ for integral values of n.

In Exercises 23 through 35 find the Fourier series representation of the given function over the indicated fundamental interval and use a computer to plot the indicated partial sum $S_n(x)$ over the fundamental interval.

23. $f(x) = \begin{cases} a, & -\pi < x < 0 \\ b, & 0 < x < \pi. \end{cases}$ Plot $S_{10}(x)$ for $a = 3, b = 1$.

24. $f(x) = \begin{cases} x+1, & -1 < x < 0 \\ x-1, & 0 < x < 1. \end{cases}$ Plot $S_{10}(x)$.

25. $f(x) = 1 - |x|$, $-1 < x < 1$. Plot $S_{10}(x)$.

26. $f(x) = \begin{cases} 0, & -2 < x < 0 \\ x, & 0 \le x < 2. \end{cases}$ Plot $S_8(x)$.

27. $f(x) = |\sin x|, -\pi \le x \le \pi$ (a fully rectified sine wave). Plot $S_{10}(x)$.

28. $f(x) = \begin{cases} ax, & -\pi < x \le 0 \\ bx, & 0 \le x < \pi. \end{cases}$ Plot $S_8(x)$ for $a = 1, b = 3$.

29. $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi. \end{cases}$ Plot $S_8(x)$.

30. $f(x) = x^2, \quad -\pi \le x \le \pi$. Plot $S_8(x)$.

31. $f(x) = x^2, \quad -2\pi \le x \le 2\pi$. Plot $S_{10}(x)$.

32. $f(x) = \sin ax, \quad -\pi \le x \le \pi$ with a not an integer. Plot $S_{10}(x)$ for $a = 0.7$.

33. $f(x) = \cos ax, \quad -\pi \le x \le \pi$ with a not an integer. Plot $S_{10}(x)$ for $a = 0.7$.

34. $f(x) = e^{ax}, \quad -\pi \le x \le \pi$. Plot $S_7(x)$ for $a = 0.7$.

35. $f(x) = \begin{cases} 0, & -2\pi \le x < -\pi \\ \sin x, & -\pi \le x \le \pi \\ 0, & \pi \le x \le 2\pi. \end{cases}$ Plot $S_8(x)$.

9.2 Convergence of Fourier Series and Their Integration and Differentiation

The general theory of the convergence of Fourier series is complicated and still incomplete in some respects. Consequently, we will only derive some useful results that can be obtained in a straightforward manner, and then state without proof a convergence theorem due to the German mathematician P. G. L. Dirichlet (1805–1859) that is sufficient for all practical applications of Fourier series.

Let us consider the nth partial sum

$$S_n(x) = a_0 + \sum_{r=1}^{n}(a_r \cos rx + b_r \sin rx), \tag{20}$$

of the Fourier series for $f(x)$ in (7) defined over the interval $-\pi \le x \le \pi$. Then, provided the integral $\int_{-\pi}^{\pi}[f(x)]^2 dx$ exists and is finite, we have the obvious result

$$\int_{-\pi}^{\pi}[f(x) - S_n(x)]^2 dx = \int_{-\pi}^{\pi}[f(x)]^2 dx - 2\int_{-\pi}^{\pi} f(x)S_n(x)dx + \int_{-\pi}^{\pi}[S_n(x)]^2 dx. \tag{21}$$

From the definition of $S_n(x)$ in (20), it follows that

$$\int_{-\pi}^{\pi}[S_n(x)]^2 dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{r=1}^{n}(a_r \cos rx + b_r \sin rx)\right]^2 dx,$$

but the orthogonality of the sine and cosine functions reduces this to

$$\begin{aligned}\int_{-\pi}^{\pi}[S_n(x)]^2 dx &= \int_{-\pi}^{\pi} a_0^2 dx + \sum_{r=1}^{n}\left[a_r^2\int_{-\pi}^{\pi}\cos^2 rx dx\right] + \sum_{r=1}^{n}\left[b_r^2\int_{-\pi}^{\pi}\sin^2 rx dx\right] \\ &= \pi\left[2a_0^2 + \sum_{r=1}^{n}(a_r^2 + b_r^2)\right].\end{aligned} \tag{22}$$

If $f(x)$ is replaced by its Fourier series, a similar argument shows that

$$\int_{-\pi}^{\pi} f(x)S_n(x)dx = \pi\left[2a_0^2 + \sum_{r=1}^{n}(a_r^2 + b_r^2)\right], \tag{23}$$

so combining (21) to (23) gives

$$\int_{-\pi}^{\pi}[f(x) - S_n(x)]^2 dx = \int_{-\pi}^{\pi}[f(x)]^2 dx - \pi\left[2a_0^2 + \sum_{r=1}^{n}(a_r^2 + b_r^2)\right]. \tag{24}$$

The integral on the left of (24) is nonnegative, because its integrand is a squared quantity, so it follows at once that for all n

$$2a_0^2 + \sum_{r=1}^{n} (a_r^2 + b_r^2) \le \frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx,$$

so letting $n \to \infty$ we arrive at the inequality

$$2a_0^2 + \sum_{r=1}^{\infty} (a_r^2 + b_r^2) \le \frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx. \tag{25}$$

Bessel's inequality

This is **Bessel's inequality** for Fourier series, and the restriction to functions $f(x)$ such that $\int_{-\pi}^{\pi}[f(x)]^2 dx$ exists and is finite implies that the series

$$2a_0^2 + \sum_{r=1}^{\infty} (a_r^2 + b_r^2)$$

is convergent, so the coefficients in the associated Fourier series (7) must be such that

$$\lim_{n\to\infty} a_n = 0 \quad \text{and} \quad \lim_{n\to\infty} b_n = 0. \tag{26}$$

the fundamental Riemann–Lebesgue lemma

This important result on the behavior of Fourier coefficients as $n \to \infty$ is called the **Riemann–Lebesgue** lemma, though its rigorous proof proceeds differently.

It is also a consequence of (24) that if the nth partial sum $S_n(x)$ converges to $f(x)$ in the sense that

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} [f(x) - S_n(x)]^2 dx = 0,$$

which is true for all functions $f(x)$ encountered in applications, then

$$2a_0^2 + \sum_{r=1}^{\infty} (a_r^2 + b_r^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx. \tag{27}$$

Parseval relation

This is the **Parseval relation** for Fourier series.

EXAMPLE 9.6

Apply the Parseval relation to the Fourier series of $f(x) = |x|$ defined over the interval $-\pi \le x \le \pi$.

Solution It follows from Example 9.4 with $L = \pi$ that the Fourier series representation of $f(x) = |x|$ over the interval $-\pi \le x \le \pi$ is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2},$$

so that

$$a_0 = \frac{\pi}{2}, \quad a_{2n-1} = -\frac{4}{\pi(2n-1)^2}, \quad \text{and} \quad a_{2n} = 0 \quad \text{for } n = 1, 2, \ldots.$$

We have

$$\int_{-\pi}^{\pi} [f(x)]^2 dx = \int_{-\pi}^{\pi} x^2 dx = \frac{2\pi^3}{3},$$

so as the integral is finite, provided $S_n(x)$ converges in the norm to $f(x)$, it follows from the Parseval relation in (27) that

$$\frac{1}{\pi}\left(\frac{2\pi^3}{3}\right) = 2\frac{\pi^2}{4} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}.$$

After simplification this reduces to the well-known result

$$\begin{aligned}\frac{\pi^4}{96} &= \sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} \\ &= \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots.\end{aligned}$$

The justification for applying the Parseval relation in this case is provided by the following theorem. It can be confirmed by summing a large number of terms and comparing the result with the known value of $\pi^4/96$. For example, using $n = 100$ leads to the result $\pi^4/96 \approx 1.01467801$, while a direct calculation shows that $\pi^4/96 = 1.01467803$, so the two results agree to seven decimal places. ■

THEOREM 9.1

Convergence of Fourier series Let $f(x)$ be continuous over the interval $-L < x < L$ except possibly at a finite number of internal points $x_1, x_2, \ldots,$ at each point x_n of which the function has a finite jump discontinuity $f(x_n+) - f(x_n-)$. Furthermore, let the left- and right-hand derivatives $f'(x_n-)$ and $f'(x_n+)$ exist for $n = 1, 2, \ldots$. Then at points of continuity of $f(x)$ its Fourier series converges uniformly to $f(x)$, and at each point of discontinuity it converges pointwise to

fundamental convergence theorem

$$\frac{1}{2}(f(x_n-) + f(x_n+)) \quad \text{for } n = 1, 2, \ldots.$$

If, in addition, $f(x)$ has a right-hand derivative $f'(-L+)$ at the left end point of the interval and a left-hand derivative $f'(L-)$ at the right end point of the interval, then at $x = \pm L$ the Fourier series converges pointwise to

$$\frac{1}{2}(f(-L+) + f(L-)).$$ ■

In effect, this theorem says that if $f(x)$ is piecewise continuous and bounded over the interval $-L < x < L$ with derivatives defined to the left and right of each discontinuity, its Fourier series converges uniformly to $f(x)$ wherever it is continuous and to the mid-point of the jump where there is a discontinuity. If, in addition, one-sided derivatives exist at the ends of the interval, then at both $x = -L$ and $x = L$ the Fourier series converges to the average of the values of $f(x)$ at the two ends of the interval.

A consequence of this theorem that is sometimes useful is that it allows many numerical series to be summed in closed form. Results of this type follow by choosing a value of x for which the terms of the Fourier series take on a simple numerical form, and equating the result to the appropriate value of $f(x)$. At a point $x = x^*$ where $f(x)$ is continuous the series will converge to $f(x^*)$, and at a point $x = x^*$ where $f(x)$ is discontinuous the series will converge to the mid-point of the jump.

EXAMPLE 9.7

(a) Given that the step function

$$f(x) = \begin{cases} -1, & \text{for } -\pi < x < 0 \\ 1, & \text{for } 0 < x < \pi \end{cases}$$

has the Fourier series

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1},$$

find a series for $\pi/4$.

(b) Given that

$$f(x) = \begin{cases} 0, & \text{for } -\pi < x < 0 \\ x^2, & \text{for } 0 \le x < \pi \end{cases}$$

has the Fourier series

$$f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \left\{ \left[\frac{2(-1)^n}{n^2} \right] \cos nx + \frac{1}{\pi} \left[(-1)^n \left(\frac{2}{n^3} - \frac{\pi^2}{n} \right) - \frac{2}{n^3} \right] \sin nx \right\},$$

find a series for $\pi^2/6$.

Solution

(a) The function $f(x)$ graphed in Fig. 9.9 is seen to be discontinuous at $x = 0$ and to have different values at $x = \pm\pi$. The average of the values of $f(x)$ to the immediate left and right of the discontinuity at $x = 0$ is zero, so the Fourier series will converge to the value zero when $x = 0$. Setting $x = 0$ in the Fourier series causes every term to vanish, so equating this to the value to which the Fourier series converges at the origin yields the uninteresting result $0 = 0$.

how Fourier series can be used to sum series

To obtain a more interesting result, let us try setting $x = \pi/2$, which makes $\sin(2n-1)\frac{\pi}{2} = (-1)^{n+1}$. The function $f(x)$ is continuous at this point and equal to 1, so its Fourier series will converge to the value 1 when $x = \pi/2$. Inserting this value of x into the Fourier series and equating the result to 1 gives

$$1 = \frac{4}{\pi} \left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \cdots \right),$$

so

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)}.$$

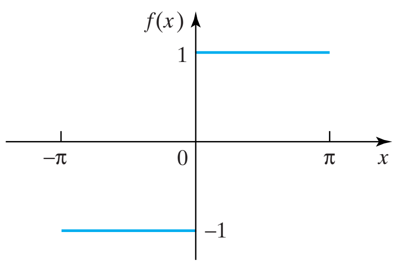


FIGURE 9.9 The step function $f(x)$.

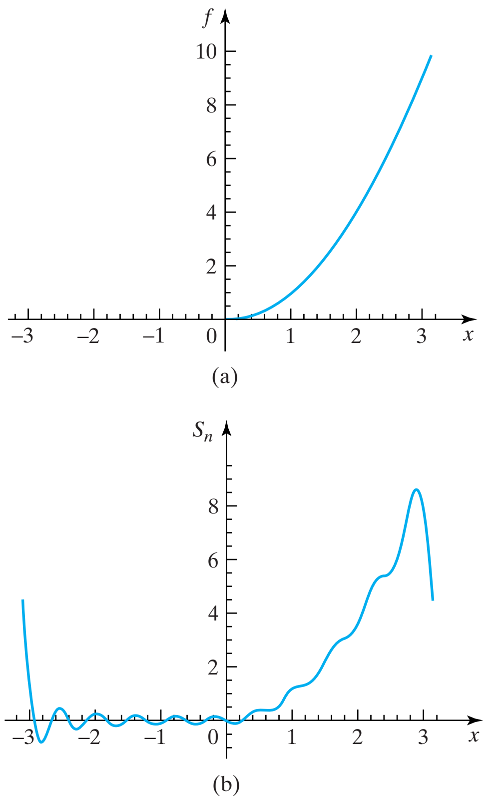


FIGURE 9.10 (a) The function $f(x)$ and (b) $S_{10}(x)$.

This series, known as Leibniz' formula, converges very slowly, so it is not useful for computing π.

(b) The function $f(x)$ is graphed in Fig. 9.10(a), and $S_{10}(x)$ in Fig. 9.10(b). The average of the values of $f(x)$ at the end points of the interval $-\pi < x < \pi$ is $\pi^2/2$, so setting $x = \pi$ in the Fourier series and equating the result to $\pi^2/2$ as required by the last part of Theorem 9.2 gives

$$\frac{\pi^2}{2} = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty}\frac{1}{n^2},$$

where we have used the fact that $\cos n\pi = (-1)^n$ and $\sin n\pi = 0$ for positive integers n.

This result simplifies to the series

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \sum_{n=1}^{\infty}\frac{1}{n^2},$$

which converges somewhat faster than the series in part (a). ■

Gibbs phenomenon

Examination of Fig. 9.3 and also Fig. 9.6 in Section 9.1 shows that when $f(x)$ is discontinuous, the graph of the partial sum $S_n(x)$ of the Fourier series representation of the function exhibits over- and undershoots close to the discontinuities. This is called the **Gibbs phenomenon**, and it persists for all values of n. This behavior

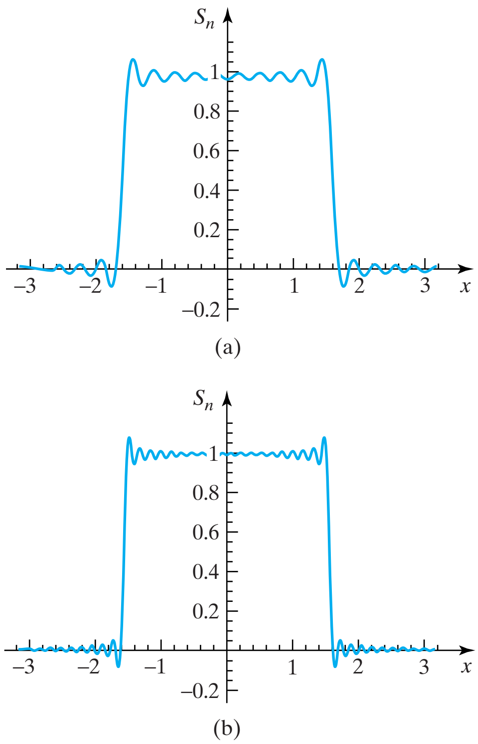


FIGURE 9.11 An example of the Gibbs phenomenon with (a) $n = 10$, and (b) $n = 20$.

reflects the way the continuous function $S_n(x)$ obtained from the Fourier series approximates the behavior of $f(x)$ at a point of discontinuity. Increasing n simply moves the under- and overshoots closer to the discontinuity while leaving their size approximately the same.

Figure 9.11 shows the Gibbs phenomena for the function

$$f(x) = \begin{cases} 0, & -\pi < x < -\pi/2 \\ 1, & -\pi/2 < x < \pi/2 \\ 0, & \pi/2 < x < \pi \end{cases}$$

for different partial sums $S_n(x)$. The results should be compared with Fig. 9.3, which shows the graph of $S_5(x)$.

We now state without proof two important theorems concerning the term-by-term integration and differentiation of Fourier series that are often useful, but before doing so we first define what are called Dirichlet conditions, which are satisfied by most functions of practical importance.

A function $f(x)$ is said to satisfy **Dirichlet conditions** on an interval $-L < x < L$ if it is bounded on the interval, has at most a finite number of maxima and minima, and is continuous apart from a finite number of discontinuities in the interval.

THEOREM 9.2

when a Fourier series can be integrated

Termwise integration of Fourier series The integral of any function $f(x)$ satisfying Dirichlet conditions on the interval $-L \le x \le L$ can be obtained by term-by-term integration of the Fourier series representation of $f(x)$. So, if $f(x)$ has the Fourier

series representation

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right) \quad \text{for } -L \le x \le L,$$

then

$$\int_{-L}^{x} f(u)du = a_0(x+L) + \frac{L}{\pi}\sum_{n=1}^{\infty}\left[\frac{a_n}{n}\sin\left(\frac{n\pi x}{L}\right) - \frac{b_n}{n}\left(\cos\left(\frac{n\pi x}{L}\right) + (-1)^{n+1}\right)\right] \quad \text{for } -L \le x \le L.$$

■

THEOREM 9.3

when a Fourier series can be differentiated

Termwise differentiation of Fourier series Let $f(x)$ be a continuous function on the interval $-L \le x \le L$ such that $f(-L) = f(L)$, and suppose also that $f'(x)$ is piecewise continuous. Then for any x strictly inside the interval at which $f''(x)$ exists, the derivative of $f(x)$ can be obtained by term-by-term differentiation of the Fourier series representation of $f(x)$. So, if $f(x)$ has the Fourier series representation

$$f(x) = a_0 + \frac{\pi}{L}\sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right) \quad \text{for } -L \le x \le L,$$

then

$$f'(x) = \frac{\pi}{L}\sum_{n=1}^{\infty}\left(-na_n \sin\left(\frac{n\pi x}{L}\right) + nb_n \cos\left(\frac{n\pi x}{L}\right)\right) \quad \text{for } -L < x < L,$$

except for points at where $f'(x)$ and $f''(x)$ are not defined. ■

EXAMPLE 9.8

Use the Fourier series representation of the function

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$

given in Example 9.7 to find a Fourier series representation of $F(x) = \int_{-\pi}^{x} f(t)dt$ in the interval $-\pi < x < \pi$, and relate the result to Example 9.4.

Solution As $f(x)$ satisfies the conditions of Theorem 9.2, its Fourier series representation may be integrated term by term to obtain the Fourier series representation of

$$F(x) = \int_{-\pi}^{x} f(t)dt = \begin{cases} \displaystyle\int_{-\pi}^{x} -1dt = -(x+\pi), & \text{for } -\pi < x < 0 \\ \displaystyle\int_{-\pi}^{0} -1dt + \int_{0}^{x} 1dt = x - \pi & \text{for } 0 < x < \pi. \end{cases}$$

From Example 9.7, the Fourier series representation of $f(x)$ is

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1},$$

so replacing x by the dummy variable t and integrating over the interval $-\pi \le t \le x$ gives

$$F(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{x}\frac{\sin(2n-1)t}{2n-1}dt = -\frac{4}{\pi}\left[\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} - \sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi}{(2n-1)^2}\right].$$

As $\cos(2n-1)\pi = -1$ for $n = 1, 2, \dots$, this reduces to

$$F(x) = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}.$$

The numerical series on the right can be summed by applying the Parseval relation to the Fourier series representation of $f(x)$ to obtain

$$2 = \sum_{n=1}^{\infty}\left(\frac{4}{\pi(2n-1)}\right)^2, \quad \text{or} \quad \frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}.$$

Replacing the numerical series in $F(x)$ by $\pi^2/8$ reduces it to

$$\int_{-\pi}^{x} f(t)dt = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} - \frac{4}{\pi}\frac{\pi^2}{8} = -\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2},$$

and so the required Fourier series representation is

$$F(x) = \left\{\begin{array}{ll} \displaystyle\int_{-\pi}^{x} -1dt = -(x+\pi), & \text{for } -\pi < x < 0 \\ \displaystyle\int_{-\pi}^{0} -1dt + \int_{0}^{x} 1dt = x - \pi, & \text{for } 0 < x < \pi \end{array}\right\}$$

$$= -\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}.$$

Examination of $F(x)$ shows that $F(x) = |x| - \pi$, so as a check we see that the Fourier series representation of the function $|x|$ in the interval $-\pi \le x \le \pi$ can be obtained by adding π to the Fourier series representation of $F(x)$ to obtain

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}, \quad \text{for } -\pi \le x \le \pi,$$

in agreement with the result of Example 9.4 with $L = \pi$. ■

EXAMPLE 9.9 Given

$$f(x) = \begin{cases} \sin 2x, & -\pi \le x < -\pi/2 \\ 0, & -\pi/2 \le x \le \pi/2 \\ \sin 2x, & \pi/2 < x \le \pi, \end{cases}$$

find $f'(x)$ by differentiation of the Fourier series representation of $f(x)$.

Solution The function satisfies the conditions of Theorem 9.3, so its Fourier series representation may be differentiated term by term to find the Fourier series representation of $f'(x)$. It was shown in Example 9.2 that the Fourier series representation of $f(x)$ is

$$f(x) = \frac{1}{2\pi} + \frac{1}{\pi}\left(\frac{2}{3}\cos x - \frac{2}{5}\cos 3x - \frac{1}{3}\cos 4x - \cdots\right)$$
$$+ \frac{1}{\pi}\left(-\frac{2}{3}\sin x + \frac{3\pi}{4}\sin 2x - \frac{2}{5}\sin 3x + \cdots\right),$$

so differentiation shows the first few terms of the Fourier series for $f'(x)$ to be

$$f'(x) = \frac{1}{\pi}\left(-\frac{2}{3}\sin x + \frac{6}{5}\sin 3x + \cdots\right) + \frac{1}{\pi}\left(-\frac{2}{3}\cos x + \frac{3\pi}{2}\cos 2x - \cdots\right),$$

where from the definition of $f(x)$

$$f'(x) = \begin{cases} 2\cos 2x, & -\pi \le x < -\pi/2 \\ 0, & -\pi/2 \le x \le \pi/2 \\ 2\cos 2x, & \pi/2 < x \le \pi. \end{cases}$$ ■

Summary

The convergence of Fourier series has been examined, and it has been shown that where $f(x)$ is continuous its Fourier series representation converges to $f(x)$, but where it has a finite jump discontinuity it converges to the mid-point of the jump. The Bessel inequality and the Parseval relation have been established, and conditions given for the termwise integration and differentiation of a Fourier series.

EXERCISES 9.2

In Exercises 1 through 4, apply the Parseval relation to the given function and its Fourier series to obtain a series representation involving a power of π.

1. $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$

with $f(x) = \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{\sin(2n-1)x}{2n-1}$.

2. $f(x) = x,\ -\pi < x < \pi$

with $f(x) = 2\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{\sin nx}{n}$.

3. $f(x) = x^2,\ -\pi \le x \le \pi$,

with $f(x) = \dfrac{\pi^2}{3} - 4\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{\cos nx}{n^2}$.

4. $f(x) = |\cos x|,\ -\pi \le x \le \pi$

with $f(x) = \dfrac{2}{\pi} + \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{\cos 2nx}{(4n^2-1)}$.

5. Show that the **Parseval relation** for a function $f(x)$ defined on the interval $-L < x < L$ takes the form

$$\frac{1}{L}\int_{-L}^{L}[f(x)]^2 dx = 2a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

6. Find the Fourier series for the function

$$f(x) = \begin{cases} 0, & -4 \le x < 0 \\ 4, & 0 \le x < 4 \end{cases}$$

and apply the Parseval relation in Exercise 5 to the result.

7. Use the Fourier series in Example 10.6(b) for the function

$$f(x) = \begin{cases} 0, & \text{for } -\pi \le x \le 0 \\ x^2, & \text{for } 0 < x < \pi \end{cases}$$

to find a series for $\pi^2/12$.

8. Use the Fourier series for $f(x) = |\sin x|$, for $-\pi \le x \le \pi$, to find a series for $\pi/4$.

9. Use the Fourier series for

$$f(x) = \begin{cases} 0, & \text{for } -1 < x < 0 \\ x, & \text{for } 0 \le x < 1 \end{cases}$$

to find a series for $\pi^2/8$.

10. Integrate the Fourier series of $f(x)$ in Exercise 2 to find the Fourier series of x^2. What happens if the Fourier series of $f(x)$ is differentiated to find $f'(x)$?

11. Find the Fourier series of $f(x) = \pi^2 - x^2$ for $-\pi \le x \le \pi$ and use it with Theorems 10.2 and 10.3 to find the Fourier series of x and $x(\pi^2 - x^2)$.

Exercises 12 through 18 are optional. Exercises 12 through 14 show how the partial sum

$$S_n(x) = a_0 + \sum_{r=1}^{n}(a_r \cos rx + b_r \sin rx),$$

of the Fourier series of a function $f(x)$ defined over the fundamental interval $-\pi \le x \le \pi$, and by periodic extension outside it, can be expressed as an integral. Exercises 15 through 17 provide an intuitive justification of Theorem 9.1.

12. Starting from the trigonometric identity

$$\frac{1}{2} + \sum_{r=1}^{n} \cos rx = \frac{\sin\left[\left(n + \frac{1}{2}\right)x\right]}{2\sin\left(\frac{x}{2}\right)}$$

that formed Exercise 19 in Section 1.4, integrate the identity first over the interval $[-\pi, 0]$ and then over the interval $[0, \pi]$ to show that

$$\int_{-\pi}^{0} \frac{\sin\left[\left(n + \frac{1}{2}\right)x\right]}{\sin\left(\frac{x}{2}\right)}dx = \pi \quad \text{and}$$

$$\int_{0}^{\pi} \frac{\sin\left[\left(n + \frac{1}{2}\right)x\right]}{\sin\left(\frac{x}{2}\right)}dx = \pi.$$

13. Substitute the Euler formulas for a_r and b_r into $S_n(x)$, after first replacing the dummy variable x in each integral by the dummy variable u to avoid confusion with the variable x in $S_n(x)$. Combine all terms under a single integral sign and, after simplifying the result using the formula $\cos a \cos b + \sin a \sin b = \cos(a - b)$, use the results of Exercise 12 to show that

$$S_n(x) = \frac{1}{\pi}\int_{x-\pi}^{x+\pi} f(x-t)\frac{\sin\left[\left(n + \frac{1}{2}\right)t\right]}{2\sin\left(\frac{t}{2}\right)}dt.$$

14. Use the periodicity of the integrand of $S_n(x)$ in Exercise 13 to show that

$$S_n(x) = \frac{1}{\pi}\int_{0}^{\pi} [f(x-t) + f(x+t)]\frac{\sin\left[\left(n + \frac{1}{2}\right)t\right]}{2\sin\left(\frac{t}{2}\right)}dt.$$

The function $D_n(t) = \sin[(n + \frac{1}{2})t]/[2\sin(\frac{t}{2})]$ occurring in the integrand of $S_n(x)$ is called the **Dirichlet kernel**.

15. Use a computer to graph $D_n(t)$ in Exercise 14 in the interval $-\pi \le t \le \pi$, for $n = 10, 15, 30$. Confirm from the graphs that when n is large $D_n(t)$ only differs significantly from zero in the interval $-2\pi/(2n+1) \le t \le 2\pi/(2n+1)$.

16. Use the conclusion of Exercise 15 together with the result

$$\int_{-\pi}^{\pi} D_n(t)dt = \pi$$

established in Exercise 12 to give reasons why for large n the Dirichlet kernel $D_n(t)$ can be approximated by the rectangular pulse function

$$\Delta(t) = \begin{cases} 0, & -\pi \le t < -2\pi(2n+1) \\ (2n+1)/4, & -2\pi/(2n+1) \le t \le 2\pi/(2n+1) \\ 0, & 2\pi/(2n+1) < t \le \pi. \end{cases}$$

17. Use the result of Exercise 16, with

$$S_n(x) = \frac{1}{\pi}\int_{0}^{\pi} [f(x-t) + f(x+t)]D_n(t)dt$$

from Exercise 14, to suggest why in the limit as $n \to \infty$ this confirms the convergence properties of Fourier series stated in Theorem 9.1.

18. By first setting $f(x) = \sin mx$ and then $f(x) = \cos mx$ in the result of Exercise 17, with m a positive integer, and using the fact that the functions $\sin mx$ and $\cos mx$ are their own Fourier series on $-\pi \le x \le \pi$, deduce that

$$\int_0^{\pi} \sin mt\, D_n(t)dt = \int_0^{\pi} \cos mt\, D_n(t)dt = \begin{cases} 0, & n = 1, 2, \ldots, m-1 \\ \pi/2, & n = m, m+1, \ldots. \end{cases}$$

9.3 Fourier Sine and Cosine Series on $0 \le x \le L$

A function $f(x)$ that is specified on the interval $0 \le x \le L$ can be represented in terms of a series either of sines or of cosines on the interval. These series are obtained by first extending the definition of the function to the interval $-L \le x \le L$ in a suitable manner, and then restricting the Fourier series representation of the extended function to the original interval $0 \le x \le L$.

Sine Series on $0 \le x \le L$

Let a function $f(x)$ specified on the interval $0 \le x \le L$ be extended to the interval $-L \le x \le L$ as an odd function by the requirement that $f(-x) = -f(x)$ for $-L \le x \le L$. Then the odd function $g(x)$ given by

$$g(x) = \begin{cases} -f(-x), & -L \le x \le 0 \\ f(x), & 0 \le x \le L, \end{cases}$$

and defined on the interval $-L \le x \le L$, coincides with the function $f(x)$ on the original interval $0 \le x \le L$.

It follows from Theorem 9.1 and the Fourier series representation of functions on the interval $-L \le x \le L$ that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad \text{for } -L \le x \le L, \tag{28}$$

where

$$b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} dx, \quad \text{for } n = 1, 2, \ldots. \tag{29}$$

As the functions $f(x)$ and $g(x)$ coincide for $0 \le x \le L$, we see that by restricting x to the interval $0 \le x \le L$, series (28) is the required sine series. Result (28) with the coefficients b_n defined by (29) is called the **sine series** representation of $f(x)$ on the interval $0 \le x \le L$, or sometimes the **half-range sine series expansion** of $f(x)$.

Cosine Series on $0 \le x \le L$

If $f(x)$ is extended to the interval $-L \le x \le L$ as an even function, by requiring that $f(-x) = f(x)$ for $-L \le x \le 0$, we can define an even function $g(x)$ by

$$g(x) = \begin{cases} f(-x), & -L \le x \le 0 \\ f(x), & 0 \le x \le L. \end{cases}$$

If we again use Theorem 9.1 with the Fourier series representation of functions on the interval $-L \le x \le L$, it follows that

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \quad \text{for } -L \le x \le L \tag{30}$$

where

$$a_0 = \frac{1}{L}\int_0^L f(x) dx \quad \text{and} \quad a_n = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi x}{L} dx, \quad \text{for } n = 1, 2, \ldots. \tag{31}$$

Here also the functions $f(x)$ and $g(x)$ coincide for $0 \le x \le L$, so by restricting x to this interval (30) is seen to provide required cosine series representation of $f(x)$ on the interval $0 \le x \le L$. Result (31) with the coefficients a_n defined by (32) is called the **cosine series** representation of $f(x)$ on the interval $0 \le x \le L$, or sometimes the **half-range cosine series expansion** of $f(x)$.

Fourier expansions only in terms of sines or cosines

Sine and cosine representations of $f(x)$ on $0 \le x \le L$

Let $f(x)$ be defined on the interval $0 \le x \le L$. Then the **sine series** representation of $f(x)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad \text{for } 0 \le x \le L,$$

where

$$b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} dx, \quad \text{for } n = 1, 2, \ldots,$$

and the **cosine series** representation of $f(x)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \quad \text{for } 0 \le x \le L,$$

where

$$a_0 = \frac{1}{L}\int_0^L f(x) dx \quad \text{and} \quad a_n = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi x}{L} dx,$$

$$\text{for } n = 1, 2, \ldots.$$

EXAMPLE 9.10 Find the sine and cosine representations of $f(x) = x$ for $0 \le x \le \pi$.

Solution The sine series representation is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx,$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx dx, \quad \text{for } n = 1, 2, \ldots.$$

Integrating this last result, we find that

$$b_n = (-1)^{n+1}\frac{2}{n},$$

so the required sine series representation is

$$f(x) = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n} \quad \text{for } 0 \le x \le \pi.$$

The cosine series representation is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx,$$

where

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x dx \quad \text{and} \quad a_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx dx \quad \text{for } n = 1, 2, \ldots.$$

Integration gives

$$a_0 = \frac{\pi}{2}, \quad \text{while } a_{2n-1} = -\frac{4}{\pi(2n-1)^2}, \quad \text{and} \quad a_{2n} = 0 \quad \text{for } n = 1, 2, \ldots,$$

so the cosine series representation is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} \quad \text{for } 0 \leq x \leq \pi.$$ ■

Summary

It has been shown how a function $f(x)$ defined on the interval $0 \leq x \leq L$ can be represented either in terms of a series involving only sine functions or as a series involving only cosine functions. These special Fourier series, called either half-range sine or cosine Fourier series, were obtained from the usual expansion over the interval $-L \leq x \leq L$ by extending the definition of $f(x)$ to the interval $-L \leq x \leq L$ in a suitable manner. As half-range Fourier series are derived from ordinary Fourier series, their convergence properties are the same as those of ordinary Fourier series.

EXERCISES 9.3

In Exercises 1 through 4 find the sine series for the given function defined on the interval $0 \leq x \leq \pi$.

1. $f(x) = x^2$.
2. $f(x) = |\cos x|$.
3. $f(x) = \begin{cases} \cos x, & 0 < x \leq \pi/2 \\ 0, & \pi/2 < x \leq \pi. \end{cases}$
4. $f(x) = (x - \pi)^2/\pi^2$.

In Exercises 5 through 8 find the cosine series for the given function defined on the interval $0 \leq x \leq \pi$.

5. $f(x) = \begin{cases} \cos x, & 0 < x \leq \pi/2 \\ 0, & \pi/2 < x \leq \pi. \end{cases}$
6. $f(x) = \sin x$.
7. $f(x) = \begin{cases} \sin x, & 0 < x \leq \pi/2 \\ 0, & \pi/2 < x \leq \pi. \end{cases}$
8. $f(x) = (x - \pi)^2/\pi^2$.
9. Use the sine series together with the orthogonality of the functions $\sin \frac{n\pi x}{L}$, for $n = 1, 2, \ldots,$ on the interval $0 \leq x \leq L$ to show that the **Parseval relation** for the **sine series** takes the form
$$\frac{2}{L}\int_0^L [f(x)]^2 dx = \sum_{n=1}^{\infty} b_n^2.$$
10. Use the cosine series together with the orthogonality of the functions $\cos \frac{n\pi x}{L}$, for $n = 1, 2, \ldots,$ on the interval $0 \leq x \leq L$ to show that the **Parseval relation** for the **cosine series** takes the form
$$\frac{2}{L}\int_0^L [f(x)]^2 dx = 2a_0^2 + 0^2 + \sum_{n=1}^{\infty} a_n^2.$$
11. Find the sine series representation of
$$f(x) = e^{-x}, \quad 0 < x < \pi.$$
12. Find the sine and cosine series representations of $f(x) = \pi - x$ on the interval $0 \leq x \leq \pi$. Use them with the results of Exercises 9 and 10 to show that
$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2} \quad \text{and} \quad \frac{\pi^4}{96} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}.$$
Comment on which series representation converges most rapidly to $f(x)$.

13.* Explain why if $f(x)$ and $g(x)$ have Fourier series representations for $-\pi \leq x \leq \pi$, the Fourier series representations of $f(x) \pm g(x)$ can be obtained from those for $f(x)$ and $g(x)$ by term-by-term addition or subtraction. By adding and subtracting the Fourier series representations of
$$\int_{-\pi}^{\pi}[f(x) + g(x)]dx \quad \text{and} \quad \int_{-\pi}^{\pi}[f(x) - g(x)]dx,$$
obtain the **generalized Parseval relation**
$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx = 2a_0 A_0 + \sum_{n=1}^{\infty}(a_n A_n + b_n B_n),$$

where the a_n, b_n are the Fourier coefficients of $f(x)$ and the A_n, B_n are the Fourier coefficients of $g(x)$.

14.* Let $f(x)$ defined for $-\pi \le x \le \pi$ be approximated by the nth partial sum of its Fourier series representation

$$S_n(x) = a_0 + \sum_{m=1}^{n} (a_m \cos mx + b_m \sin mx),$$

and let

$$\Phi(x) = A_0 + \sum_{m=1}^{n} (A_m \cos mx + B_m \sin mx)$$

be any other approximation to $f(x)$ with coefficients A_m and B_m. Show by expanding the square error

$$E_n = \int_{-\pi}^{\pi} [f(x) - \Phi_n(x)]^2 dx$$

in terms of the Fourier series representation of $f(x)$ that E_n is minimized when $A_m = a_m$ and $B_m = b_m$ for $m = 0, 1, 2, \ldots, n$. This establishes the fact that the Fourier series partial sum $S_n(x)$ provides the best trigonometric approximation to $f(x)$ in the least squares sense.

9.4 Other Forms of Fourier Series

In this section we introduce two other forms of Fourier series that prove useful. The first is the Fourier series of a function $f(x)$ defined over an interval $a - L \le x \le a + L$ with a an arbitrary real number, and by periodicity outside it. Frequently $a = L$, corresponding to the Fourier series over the interval $0 \le x \le 2L$. The second form of Fourier series considered uses the Euler identity $e^{ix} = \cos x + i \sin x$ to derive the **complex** form of the Fourier series, also often called the **exponential form** of the Fourier series.

Fourier Series over a Shifted Interval

Routine integration shows the set of functions

$$1, \quad \sin\frac{n\pi x}{L} \quad \text{and} \quad \cos\frac{n\pi x}{L} \quad \text{for } n = 1, 2, \ldots$$

form an orthogonal system over any interval of the form $a - L \le x \le a + L$, for any real number a, and that

$$\int_{a-L}^{a+L} \sin\frac{m\pi x}{L} \cos\frac{n\pi x}{L} dx = 0 \quad \text{for all integers } m \text{ and } n,$$

$$\int_{a-L}^{a+L} \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L} dx = \begin{cases} 0 & \text{for } m \ne n \\ L & \text{for } m = n, \text{ for all integers } m \text{ and } n, \end{cases}$$

$$\int_{a-L}^{a+L} \cos\frac{m\pi x}{L} \cos\frac{n\pi x}{L} dx = \begin{cases} 0 & \text{for } m \ne n \\ L & \text{for } m = n \ne 0 \\ 2L & \text{for } m = n = 0, \text{ for all integers } m \text{ and } n. \end{cases}$$

The following result is a direct consequence of these integrals, and it provides an extension of the definition of a Fourier series to the interval $-L \le x \le L$.

Fourier series over a shifted interval

Fourier series over the interval $a - L \le x \le a + L$

A function $f(x)$ defined on the interval $a - L \le x \le a + L$ has the Fourier series representation

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right), \tag{32}$$

where

$$a_0 = \frac{1}{2L}\int_{a-L}^{a+L} f(x)dx, \quad a_n = \frac{1}{L}\int_{a-L}^{a+L} f(x)\cos\frac{n\pi x}{L}dx, \tag{33}$$
$$b_n = \frac{1}{L}\int_{a-L}^{a+L} f(x)\sin\frac{n\pi x}{L}dx, \quad \text{for } n = 1, 2, \ldots .$$

EXAMPLE 9.11 Find the Fourier series representation of

$$f(x) = \begin{cases} x, & 0 \le x \le \pi \\ \pi, & \pi \le x < 2\pi. \end{cases}$$

Solution A graph of the function $f(x)$ is shown in Fig. 9.12. Using (33) with $a = L = \pi$ gives

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx = \frac{3\pi}{4} \quad \text{and} \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx,$$

from which it follows that

$$a_{2n-1} = -\frac{2}{\pi(2n-1)^2} \quad \text{and} \quad a_{2n} = 0 \quad \text{for } n = 1, 2, \ldots .$$

The Euler formula for b_n gives

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx = -\frac{1}{n} \quad \text{for } n = 1, 2, \ldots,$$

so the required Fourier series is

$$f(x) = \frac{3\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} - \sum_{n=1}^{\infty}\frac{\sin nx}{n} \quad \text{for } 0 \le x < 2\pi. \quad \blacksquare$$

Complex Fourier Series

The Euler identities $e^{ix} = \cos x + i\sin x$ and $e^{-ix} = \cos x - i\sin x$ allow us to write

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \text{and} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

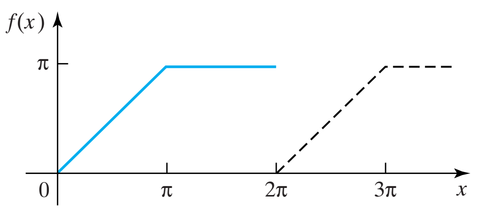


FIGURE 9.12 The function $f(x)$ defined for $0 \le x < 2\pi$.

When these results are used in the real variable Fourier series representation of $f(x)$ over the interval $-L \le x \le L$, it becomes

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \left(\frac{e^{in\pi x/L} + e^{-in\pi x/L}}{2} \right) + b_n \left(\frac{e^{in\pi x/L} - e^{-in\pi x/L}}{2i} \right) \right],$$

and after grouping terms we have

$$f(x) = a_0 + \sum_{n=1}^{n} \left(\frac{a_n - ib_n}{2} \right) e^{in\pi x/L} + \sum_{n=1}^{n} \left(\frac{a_n + ib_n}{2} \right) e^{-in\pi x/L}. \tag{34}$$

If we now define

$$c_0 = a_0, \quad c_n = \frac{a_n - ib_n}{2}, \quad \text{and} \quad c_{-n} = \frac{a_n + ib_n}{2} \quad \text{for } n = 1, 2, \ldots, \tag{35}$$

the Fourier series representation of $f(x)$ in (34) becomes

$$f(x) = \lim_{k\to\infty} \sum_{n=-k}^{k} c_n e^{in\pi x/L} \quad \text{for } -L \le x \le L. \tag{36}$$

This is the **complex** or **exponential** form of the Fourier series representation of $f(x)$.

If real functions $f(x)$ are considered, the Fourier coefficients a_n and b_n are real, and (35) then shows that c_n and c_{-n} are complex conjugates, because $c_{-n} = \bar{c}_n$. To proceed further we now make use of the fact that the functions $\exp(im\pi x/L)$ and $\exp(-in\pi x/L)$ are orthogonal over the interval $-L \le x \le L$, because integration shows that

$$\int_{-L}^{L} e^{im\pi x/L} e^{-in\pi x/L} dx = \begin{cases} 0, & \text{for } m \ne -n \\ 2\pi & \text{for } m = -n \text{ for } m, n \text{ positive integers.} \end{cases}$$

Multiplication of (36) by $\exp(-im\pi x/L)$, followed by integration over $-L \le x \le L$ and use of the above orthogonality condition gives

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-in\pi x/L} dx, \quad \text{for } n = 0, \pm 1, \pm 2, \ldots. \tag{37}$$

Collecting these results we arrive at the following definition.

the complex or exponential form of a Fourier series

The complex form of a Fourier series

Let the real function $f(x)$ be defined on the interval $-L \le x \le L$. Then the complex Fourier series representation of $f(x)$ is

$$f(x) = \lim_{k\to\infty} \sum_{n=-k}^{k} c_n e^{in\pi x/L} \quad \text{for } -L \le x \le L,$$

where

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-in\pi x/L} dx, \quad \text{for } n = 0, \pm 1, \pm 2, \ldots.$$

As the complex form of a Fourier series was derived directly from the real variable Fourier series, it follows directly that if $f(x)$ is defined for $a - L \le x \le a + L$, then

$$f(x) = \lim_{k\to\infty} \sum_{n=-k}^{k} c_n e^{in\pi x/L} \quad \text{for } a - L \le x \le a + L, \tag{38}$$

with

$$c_n = \frac{1}{2L}\int_{a-L}^{a+L} f(x)e^{-in\pi x/L}dx, \quad \text{for } n = 0, \pm 1, \pm 2, \ldots. \tag{39}$$

It is sometimes useful to separate out the coefficient c_0 from the summation in (36) (or in (38)) by writing

$$f(x) = c_0 + \lim_{k\to\infty} \sum_{n=-k}^{k}{}' c_n e^{in\pi x/L}, \tag{40}$$

with the understanding that Σ' indicates that the term corresponding to $n = 0$ has been omitted from the summation.

When $f(x)$ is real, so that $c_{-n} = \bar{c}_n$, result (40) becomes

$$f(x) = c_0 + \sum_{n=1}^{\infty}[c_n e^{in\pi x/L} + \bar{c}_n e^{-in\pi x/L}]. \tag{41}$$

Because the complex form of the Fourier series representation of a function is derived from its real variable definition, the convergence properties of complex Fourier series are the same as those already discussed for the real variable case. So at points of continuity of $f(x)$ the complex Fourier series converges uniformly to $f(x)$, while at points of discontinuity it converges to the mid-point of the jump discontinuity.

EXAMPLE 9.12

Find the complex Fourier series representation of

$$f(x) = \begin{cases} 0, & -\pi < x < -\pi/2 \\ 1, & -\pi/2 < x < \pi/2 \\ 0, & \pi/2 < x < \pi. \end{cases}$$

Solution As the function $f(x)$ is defined on the interval $-\pi \le x \le \pi$, we have $L = \pi$, so the coefficients c_n are given by

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 1dx = \frac{1}{2}$$

and

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{-inx}dx = \frac{1}{n\pi}\left(\frac{e^{in\pi/2} - e^{-in\pi/2}}{2i}\right)$$

$$\text{for } n = \pm 1, \pm 2, \ldots.$$

The coefficients c_n reduce to the real values

$$c_n = \frac{1}{n\pi}\sin\frac{n\pi}{2} \quad \text{for } n = \pm 1, \pm 2, \ldots,$$

so $c_n = c_{-n}$ because c_n is an even function of n. Consideration of the function

$\sin(n\pi/2)$ for integer values of n shows that

$$c_{2n-1} = \frac{(-1)^{n-1}}{\pi(2n-1)} \quad \text{and} \quad c_{2n} = 0 \quad \text{for } n = 1, 2, \ldots.$$

Thus, the complex Fourier series representation of $f(x)$ is

$$f(x) = \frac{1}{2} + \lim_{k\to\infty} {\sum_{n=-k}^{k}}' c_n(e^{inx} + e^{-inx}).$$

The real variable Fourier series representation of this function $f(x)$ was derived in Chapter 8, Example 8.22, and considered again at the start of Section 9.1. If c_n is used in the preceding result with $e^{inx} + e^{-inx} = 2\cos nx$, the complex Fourier series representation reduces to the real variable Fourier series representation

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(2n-1)x}{(2n-1)}$$

that was obtained previously. This series, and the equivalent complex series, converges uniformly to $f(x)$ at points of continuity of $f(x)$ and to the value $1/2$ at the discontinuities located at $x = \pm\pi/2$. ■

EXAMPLE 9.13 Find the complex Fourier series representation of

$$f(x) = \begin{cases} 0, & 0 < x < 1 \\ 1, & 1 < x < 4. \end{cases}$$

Solution The function $f(x)$ is defined on the interval $0 \le x \le 2L$, with $2L = 4$, so $L = 2$. Thus, the complex Fourier coefficients c_n are given by

$$c_n = \frac{1}{4}\int_0^4 f(x)e^{-in\pi x/2}dx = \frac{1}{4}\int_1^4 e^{-in\pi x/2}dx, \quad \text{for } n = 0, \pm 1, \pm 2, \ldots.$$

Setting $n = 0$ gives

$$c_0 = \frac{3}{4},$$

whereas

$$c_n = \frac{i}{2\pi n}\left[1 - e^{-in\pi/2}\right], \quad \text{for } n = \pm 1, \pm 2, \ldots.$$

So the complex Fourier series representation of $f(x)$ is

$$f(x) = c_0 + \lim_{k\to\infty}\sum_{n=-k}^{k} c_n e^{in\pi x/2},$$

with c_0 and c_n defined as shown. ■

Accounts of Fourier series and their general properties are to be found in references [3.3] to [3.5] and also in [3.7], [3.16], and [4.2]. An advanced and encyclopedic account of trigonometric series is given in reference [4.5].

Summary

Other forms of Fourier series have been derived, first by stretching and shifting the interval over which the expansion was required, and then by expressing the series in complex form. As both results were derived from the ordinary Fourier series, their convergence properties are the same as those of ordinary Fourier series.

EXERCISES 9.4

In Exercises 1 through 4 find the Fourier series representation of the function $f(x)$ over the given shifted interval.

1. $f(x) = \begin{cases} 0, & 0 < x < \pi \\ 1, & \pi < x < 2\pi. \end{cases}$
2. $f(x) = 1 - x, \quad 0 < x < 1.$
3. $f(x) = x, \quad 0 < x < \pi.$
4. $f(x) = x^2, \quad \pi < x < 3\pi.$

In Exercises 5 through 10 find the complex Fourier series representations of the given function $f(x)$ over the stated interval.

5. $f(x) = e^x, \quad -1 < x < 1.$
6. $f(x) = x^2, \quad 0 < x < 2\pi.$
7. $f(x) = e^x, \quad 0 < x < 1.$
8. $f(x) = \sinh x, \quad -\pi < x < \pi.$
9. $f(x) = e^x, \quad -\pi < x < \pi.$
10. $f(x) = \cosh x, \quad -1 < x < 1.$

9.5 Frequency and Amplitude Spectra of a Function

When Fourier series are applied to periodic physical phenomena with period T, it is convenient to work in terms of the angular frequency ω_0 defined as

$$\omega_0 = \frac{2\pi}{T}, \tag{42}$$

where $1/T = \omega_0/2\pi$ measures the number of cycles (oscillations) occurring in one time unit. For example, the period of the function $\sin 2x$ is $T = \pi$, so in this case $\omega_0 = 2$.

interpreting Fourier series representations in a different way

The Fourier series representation of a function $f(x)$ defined on the interval $-L \le x \le L$ with the corresponding period $T = 2L$ has been shown to be

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right),$$

so as $\omega_0 = \pi/L$ this can be written

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 x + b_n \sin n\omega_0 x), \tag{43}$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = \frac{1}{L}\int_{-L}^{L} f(x)\cos n\omega_0 x dx \quad \text{for } n = 1, 2, \ldots, \tag{44}$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = \frac{1}{L}\int_{-L}^{L} f(x)\sin n\omega_0 x dx \quad \text{for } n = 1, 2, \ldots. \tag{45}$$

In terms of these results (43) becomes

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)^{1/2} \left[\frac{a_n}{(a_n^2 + b_n^2)^{1/2}} \cos n\omega_0 x + \frac{b_n}{(a_n^2 + b_n^2)^{1/2}} \sin n\omega_0 x \right]. \tag{46}$$

Using the trigonometric identity $\cos(P + Q) = \cos P \cos Q - \sin P \sin Q$, and defining

$$A_n = (a_n^2 + b_n^2)^{1/2} \quad \text{and} \quad \delta_n = \operatorname{Arctan}(-b_n/a_n), \tag{47}$$

with A_n the **amplitude** and δ_n the **phase**, allows (46) to be written more concisely in the **amplitude and phase angle** representation

$$f(x) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 x + \delta_n). \tag{48}$$

When the Fourier series representation of $f(x)$ is expressed in this form, the set of numbers

$$\omega_0, \quad 2\omega_0, \quad 3\omega_0, \ldots$$

frequency spectrum, amplitude, and phase

is called the **frequency spectrum** of the function $f(x)$. The number $n\omega_0$ is called the ***n*th harmonic frequency** of $f(x)$, and the number δ_n the ***n*th phase angle** of $f(x)$. The set of numbers

$$A_0, \quad A_1, \quad A_2, \ldots,$$

where $A_0 = |a_0|$, is called the **amplitude spectrum** of $f(x)$, and the function

$$\cos(n\omega_0 x + \delta_n)$$

is called the ***n*th harmonic** of the function $f(x)$. The amplitude spectrum can be displayed graphically by drawing lines of height $A_0, A_1, A_2, \ldots,$ against the respective harmonic frequencies $\omega_0, 2\omega_0, 3\omega_0, \ldots,$ as shown in the next example. This is called a **discrete spectrum**, because the amplitude is only defined at the discrete frequencies in the frequency spectrum.

Result (48) shows how $f(x)$ is representable in terms of a linear combination of harmonics, each weighted by an appropriate amplitude factor A_n.

EXAMPLE 9.14 Find the harmonics and amplitude spectrum of

$$f(x) = \begin{cases} \pi, & -\pi < x < 0 \\ \pi - x, & 0 \le x \le \pi. \end{cases}$$

Solution In this case the function is defined on the interval $-\pi \le x \le \pi$, so $L = \pi$, $T = 2L = 2\pi$, and $\omega_0 = 2\pi/T = 1$. The frequency spectrum becomes $1, 2, 3, \ldots,$

and the Fourier series representation in terms of frequency is

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{0} \pi\, dx + \frac{1}{2\pi}\int_{0}^{\pi}(\pi - x)dx = \frac{3\pi}{4},$$

and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} \pi \cos nx dx + \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\cos nx dx = \frac{1}{\pi n^2}[1-(-1)^n], \quad \text{for } n = 1, 2, \ldots.$$

This last result simplifies to

$$a_{2n-1} = \frac{2}{\pi(2n-1)^2}, \quad a_{2n} = 0, \quad \text{for } n = 1, 2, \ldots.$$

Similarly,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} \pi \sin nx dx + \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\sin nx dx = \frac{(-1)^n}{n}, \quad \text{for } n = 1, 2, \ldots.$$

Substituting the coefficients a_n and b_n into the Fourier series gives

$$f(x) = \frac{3\pi}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} + \sum_{n=1}^{\infty}\frac{(-1)^n \sin nx}{n} \quad \text{for } -\pi \le x \le \pi.$$

To find the harmonics and the amplitude spectrum, it is necessary to group together terms with corresponding frequencies. When this is done $f(x)$ becomes

$$f(x) = \frac{3\pi}{4} + \left(\frac{2}{\pi}\cos x - \sin x\right) + \frac{1}{2}\sin 2x + \left(\frac{2}{9\pi}\cos 3x - \frac{1}{3}\sin 3x\right)$$
$$+ \frac{1}{4}\sin 4x + \left(\frac{2}{25\pi}\cos 5x - \frac{1}{5}\sin 5x\right) + \cdots.$$

This shows, for example, that the fifth harmonic is proportional to

$$\frac{2}{25\pi}\cos 5x - \frac{1}{5}\sin 5x.$$

The amplitudes are

$$A_0 = |a_0| = \frac{3\pi}{4}, \quad A_1 = \left[\left(\frac{2}{\pi}\right)^2 + (-1)^2\right]^{1/2},$$
$$A_2 = \frac{1}{2}, \quad A_3 = \left[\left(\frac{2}{9\pi}\right)^2 + \left(-\frac{1}{3}\right)^2\right]^{1/2},$$
$$A_4 = \frac{1}{4}, \quad A_5 = \left[\left(\frac{2}{25\pi}\right)^2 + \left(-\frac{1}{5}\right)^2\right]^{1/2}, \ldots.$$

In general

$$A_{2n-1} = \frac{1}{(2n-1)}\left[\frac{4}{(2n-1)^2\pi^2} + 1\right]^{1/2} \quad \text{and} \quad A_{2n} = \frac{1}{2n}, \quad \text{for } n = 1, 2, \ldots.$$